Sample Exam 1 Key **Math 221H** **Fall 2008**

Name:____________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. No electronic equipment is allowed.

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(16) **1.** Specify the order of the following ODEs and classify (do *not* solve) the first order ODEs as autonomous(A), exact(E), linear(L) or separable(S), if any. Also indicate what method you would use to solve them (do not actually do it.)

(a) $y\,y' = x\left(y^2+1\right)$

Solution. First order, (S), use separation of variables.

(b) $yy'' + (y')^2 = 0$

Solution. Second order, use reduction of order substitution $u = y'$, $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$.

(c) $\dfrac{dy}{dx} - \dfrac{y}{2x} = 2xy^{-1}$

Solution. First order, use substitution for homogeneous equation $u = y/x$.

(d) $\left(3x^2+2y^2\right) + \left(4xy+6y^2\right)\dfrac{dy}{dx} = 0$

Solution. First order, (E), find function for which this is differential.

(14) **2.** A tank contains 1000 liters (L) of a solution of 100 kg of salt dissolved in water. Pure water is pumped in at a rate of 5 L/s and the stirred solution is pumped out at the same rate. Assume that $S(0) = 100$, find a differential equation condition for the amount $S(t)$ of salt in the tank at time t and solve for S.

Solution. Volume is constant 1000. At time t we have the amount of entering salt is 0 kg/s and the amount of exiting salt is $5\cdot\dfrac{1}{1000}S$ kg/s. Hence the change in salt over time interval dt is

$$\begin{aligned} dS &= \text{Salt in} - \text{Salt out} = \text{Rate in}\cdot dt - \text{Rate out}\cdot dt \\ &= 0dt - \frac{S}{200}dt, \end{aligned}$$

so that $dS/dt = -\dfrac{S}{200}$ and so $\dfrac{dS}{S} = -\dfrac{dt}{200}$ and integrating gives

$$\ln S = \int \frac{dS}{S} = \int -\frac{dt}{200} = -\frac{t}{200} + C.$$

Take exponentials of both sides and

$$S(t) = e^{\ln S} = e^{-\frac{t}{200}}e^C,$$

so that $S(0) = 100 = e^0 e^C$ and $e^C = 100$. Thus $S(t) = 100e^{-\frac{t}{200}}$.

(27) **3.** Find all solutions to these DEs or IVPs:

(a) $\dfrac{dy}{dx} = y \sin x$

SOLUTION. Separate variables by multiplying both sides by dx/y, so that $\dfrac{dx}{y}\left\{\dfrac{dy}{dx} = y \sin x\right\}$ gives $\dfrac{dy}{y} = \sin x \, dx$ and integrating both sides gives

$$\ln|y| = \int \frac{dy}{y} \quad = \quad \int \sin x \, dx = -\cos x + C.$$

Take exponentials of both sides and obtain

$$e^{\ln|y|} = |y| \quad = \quad e^{-\cos x + C} = e^{C} e^{-\cos x},$$

so that $y = \pm e^{C} e^{-\cos x} = A e^{-\cos x}$ with A an arbitrary constant.

(b) $(4x - y) + (6y - x)\dfrac{dy}{dx} = 0$

SOLUTION. Multiply by dx to write the equation as $(4x - y)\,dx + (6y - x)\,dy = 0$. This equation is exact since $\partial/\partial y(4x - y) = -1 = \partial/\partial x(6y - x)$. So this is an exact differential dF, with $F_x = 4x - y$ and $F_y = 6y - x$.

Thus $F = \int F_x dx = \int (4x - y)\,dx = 2x^2 - yx + C(y)$. Now differentiate with respect to y and obtain that $F_y = 6y - x = -x + C'(y)$, so that $C'(y) = 6y$ and $C(y) = \frac{6y^2}{2} = 3y^2$. Thus the solution to the DE is

$$F(x, y) = 2x^2 - yx + 3y^2 = D,$$

where D is an arbitrary constant.

(c) $y' + 3y = 2xe^{-3x}, \quad y(0) = 2$

SOLUTION. This equation is linear. An integrating factor is $\rho = e^{\int 3dx} = e^{3x}$. Multiply the equation by ρ and obtain $e^{3x}y' + 3e^{3x}y = (e^{3x}y)' = 2xe^{-3x}e^{3x} = 2x$. Thus $e^{3x}y = 2\frac{x^2}{2} + C = x^2 + C$ for some constant C. Plug in $y(0) = 2$ and obtain $e^0 2 = 0 + C$, so $C = 2$ and

$$y = e^{-3x}\left(x^2 + 2\right)$$

(12) **4.** The acceleration of a Maserati is proportional to the difference between 250 km/hr and the velocity of this sports car. Set up a DE for velocity of this car, sketch its phase diagram and sketch a few representative solutions to the DE.

SOLUTION. Since acceleration is dv/dt, where $v(t)$ is velocity at time t, the assertion about acceleration says that

$$\frac{dv}{dt} = k(250 - v),$$

where the constant of proportionality k must be positive (otherwise acceleration would be negative and the car would go backwards unless velocity could exceed 250 km/hr.) Therefore the critical points of this DE are just where $0 = f(v) = k(250 - v)$, i.e., $v = 250$. Furthermore, the rhs is positive for $v < 250$ and negative for $v > 250$. So the phase diagram and graphs of typical solutions look like

(16) **5.** You are given the DE $\frac{dy}{dx} = \sqrt[3]{y}$.

(a) Find the equilibrium solutions to this DE and classify them as stable or unstable.

SOLUTION. The equilibrium solutions to this DE are given by $0 = f(y) = \sqrt[3]{y}$. tjat is, $y(x) = 0$. Since $f < 0$ on the left of 0 and $f > 0$ on the right of zero, solutions push away from the zero solution, so this is an unstable equilibrium point.

(b) Determine the extent to which the existence/uniqueness theorem applies to the given DE with initial condition $y(0) = 0$.

SOLUTION. The function $f(y)$ is continuous for all y, so certainly in any box in the xy-plane containing the point $(0, 0)$. Therefore the existence theorem guarantees that a solution to the IVP with $y(0) = 0$ exists.

However, $f_y(y) = (1/3)\, y^{-2/3}$ is not continuous at $y = 0$, and therefore uniqueness of this solution is not guaranteed.

(15) **6.** A population is modeled by the autonomous differential equation $\frac{dx}{dt} = 4x - x^2 - h$ with parameter h. Sketch the bifurcation diagram for this DE and label the critical points as stable or unstable. Sketch some typical solutions to the DE for $h = 2$.

SOLUTION. Critical points are given by $f(x) = 4x - x^2 - h = 0$, i.e., $h = 4x - x^2 = (4 - x)\, x$. The vertex is at $h = 4$ and this parabola crosses the x-axis at $x = 4, 0$, so the bifurcation diagram looks like

Also, at $h = 2$, $4x - x^2 = 2$ and $x^2 - 4x + 2 = 0$, so $x = \left(4 \pm \sqrt{16 - 8}\right)/2 = 2 \pm \sqrt{2}$. Fill in a phase line on the vertical x-axis noting that $4x - x^2 - 2$ is negative for $x > 2 + \sqrt{2}$, positive for $2 - \sqrt{2} < x < 2 + \sqrt{2}$, and negative for $x < 2 - \sqrt{2}$. So we get a graph of typical solutions that looks like