Quiz 6 **Math 208** **Fall 2009**

Name:____________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14), and you should make obvious simplifications. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is optional.

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(10) **1.** Set up (do not evaluate) an iterated integral for $\iint_S G(x,y,z)\,d\sigma$, where $G(x,y,z) = x^2$ and S is the unit sphere $x^2+y^2+z^2=1$.

SOLUTION. In spherical coordinates the surface S is $\rho = 1$. Parametrize the sphere with ϕ and θ of spherical coordinates and $\rho = 1$, so the position vector and cross product are

$$\begin{aligned}
\mathbf{r}(x(\phi,\theta), y(\phi,\theta), z(\phi,\theta)) &= \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\
&= \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos^2\theta\sin\phi\cos\phi + \sin^2\theta\sin\phi\cos\phi \right\rangle \\
&= \sin\phi\,\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle
\end{aligned}$$

so that

$$\begin{aligned}
d\sigma &= |\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA \\
&= \sin\phi\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi}dA \\
&= \sin\phi dA
\end{aligned}$$

Thus

$$\iint_S x^2 d\sigma = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta\sin\phi\,d\phi\,d\theta.$$

(10) **2.** State and use Stokes' Theorem to set up as an iterated integral (do not evaluate) the circulation (flow) of $\mathbf{F}(x,y,z) = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$ around the boundary C of the triangle cut from the plane $x+y+z=1$ by the first octant, oriented counterclockwise when viewed from above.

SOLUTION. Stokes' Theorem says that if the orientable surface S has boundary curve C positively oriented with respect to the normal vector filed $\mathbf{n}$ for S, then $\int_C \mathbf{F}\cdot\mathbf{T}ds = \iiint_D (\nabla\times\mathbf{F})\cdot\mathbf{n}d\sigma$. Parametrize the surface S with x and y. Then the surface is the graph of $z = f(x,y) = 1-x-y$ over the region R in the xy-plane bounded by the x-axis, y-axis and line $x+y=1$. Thus we have $\mathbf{n}d\sigma = \pm\langle 1,1,1\rangle\,dA$, where we choose the plus sign to get the right orientation, and thus

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = \langle 0-x, -(2x-0), z-1\rangle = \langle -x, -2x, z-1\rangle$$

so that

$$\begin{aligned}
\nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma &= \langle -x,-2x,z-1\rangle\cdot\langle 1,1,1\rangle\,dA \\
&= (-x-2x+(1-x-y)-1)\,dA \\
&= (-4x-y)\,dA.
\end{aligned}$$

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^1\int_0^{1-x}(-4x-y)\,dy\,dx.$$