(24) 1. Let a surface \( S \) be given by \( z = 2x + y^2 \) where \((x, y) \in R = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 4\}\).

(a) Find formulas for vector and scalar differential surface area \( n\,d\sigma \) and \( d\sigma \) in terms of \( dA \), differential surface area in the \( xy \)-plane, where \( n \) is the downward oriented normal.

SOLUTION. The surface is the graph of \( f(x, y) = 2x + y^2 \), so we have

\[
 n\,d\sigma = \pm (-f_x, -f_y, 1) \, dA = -(-2, -2y, 1) \, dA = \langle 2, 2y, -1 \rangle \, dA,
\]

where we choose the minus sign for the correct orientation of \( n \). Therefore

\[
 d\sigma = |n\,d\sigma| = |n| \, d\sigma = \sqrt{4 + 4y^2 + 1} \, dA = \sqrt{5 + 4y^2} \, dA.
\]

(b) Express \( \iint_S f(x, y, z) \, d\sigma \) as an iterated integral in \( x \) and \( y \) where \( f(x, y, z) = \sin(xy^2 \! z) \). Do not work the integral out.

SOLUTION.

\[
 \iint_S f(x, y, z) \, d\sigma = \iint_R f(x, y, 2x + y^2) \, \sqrt{5 + 4y^2} \, dA = \int_0^3 \int_0^4 f(x, y, 2x + y^2) \, \sqrt{5 + 4y^2} \, dy \, dx
\]

(c) Calculate \( \iint_S \mathbf{F} \cdot n \, d\sigma \), where \( \mathbf{F} = \langle 2z, 0, 4y^2 \rangle \).

SOLUTION.

\[
 \iint_S \mathbf{F} \cdot n \, d\sigma = \iint_R \langle 2z, 0, 4y^2 \rangle \, \langle 2, 2y, -1 \rangle \, dA
\]

\[
 = \int_0^3 \int_0^4 (4z - 4y^2) \, dy \, dx
\]

\[
 = \int_0^3 \int_0^4 (4(2x + y^2) - 4y^2) \, dy \, dx
\]

\[
 = \int_0^3 \int_0^4 8xy \, dx \, dy
\]

\[
 = \frac{8x^2}{2} \bigg|_0^3 \bigg| y \bigg|_0^4 = 36 \cdot 4 = 144.
\]
(16) 2. Apply the flow form of Green’s Theorem to compute the area of the ellipse \( x^2/4 + y^2/9 = 1 \) by using \( F = x \mathbf{j} \). (It may help to recall that this ellipse can be parametrized by \( x = 2 \cos t \), \( y = 3 \sin t \).)

**SOLUTION.** The flow form of Green’s Theorem is, with \( \mathbf{T} \cdot d\mathbf{s} = (dx, dy) \),
\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C M \, dx + N \, dy = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA = \iint_R (N_x - M_y) \, dA,
\]
and in our case \( M = 0 \) and \( N = x \), so that \( N_x - M_y = 1 \) and the area of \( R \) is
\[
\iint_R 1 \, dA = \iint_R (N_x - M_y) \, dA = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x \, dy = \int_0^{2\pi} 2 \cos t \cos t \, dt,
\]
since the parametrization of \( C \) in the hint implies that \( dy = 3 \cos t \, dt \). Thus then area of \( R \) is
\[
6 \int_0^{2\pi} \cos^2 t \, dt = 6 \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = 6 \left( \frac{1}{2}t - \frac{\sin 2t}{4} \right) \bigg|_0^{2\pi} = 6 \left( \frac{2\pi}{2} - 0 \right) = 6\pi.
\]

(26) 3. Let \( \mathbf{F} = (yz, xz, xy) \).

(a) Show that \( \mathbf{F} \) is conservative without actually finding a potential function for \( \mathbf{F} \).

**SOLUTION.** Calculate the curl of \( \mathbf{F} \):
\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \left( \frac{\partial}{\partial y} xy - \frac{\partial}{\partial z} xz, -\left( \frac{\partial}{\partial x} xy - \frac{\partial}{\partial z} yz \right), \frac{\partial}{\partial x} xz - \frac{\partial}{\partial y} yz \right) = (x-x, -(y-y), x-x) = (0,0,0).
\]

Since the curl is zero in a simply connected region (3D space), \( \mathbf{F} \) is a conservative vector field.

(b) Find a potential function \( f(x, y, z) \) for \( \mathbf{F} \).

**SOLUTION.**
Say \( \mathbf{F} = (f_x, f_y, f_z) \), so that \( f_x = yz \), \( f_y = xz \) and \( f_z = xy \). Then

\[
f = \int f_x \, dx = yz + C(y, z), \quad f_y = xz + C_y(y, z) = xz, \quad C(y, z) = \int C_y \, dy = \int 0 \, dy = D(z).
\]

This implies that \( f = xyz + D(z) \), so that \( f_x = xy + D'(z) = xy \), \( f_y = xz + D_y(y, z) = xz \), so that \( D'(z) = 0 \) and we can take \( D = 0 \). Hence a potential function for \( \mathbf{F} \) is
\[
f(x, y, z) = xyz.
\]

(c) Find the value of the line integral \( \int_C yz \, dx + xz \, dy + xy \, dz \), where \( C \) is the curve given by position vector \( \mathbf{r} = \langle t, \cos t, e^t \rangle \), \( 0 \leq t \leq \pi \).

**SOLUTION.** This line integral is just \( \int_C \mathbf{F} \cdot d\mathbf{r} \), with \( \mathbf{F} = (yz, xz, xy) \) and \( d\mathbf{r} = (dx, dy, dz) \). Therefore, we use the potential function of (b) to obtain
\[
\int_C yz \, dx + xz \, dy + xy \, dz = \int_C \mathbf{F} \cdot d\mathbf{r} = f(\pi, \cos \pi, e^\pi) - f(0, \cos 0, e^0) = \pi (-1) e^\pi - 0 \cdot 1e^0 = -\pi e^\pi.
\]
(20) 4. Let $C$ be the boundary of the triangle $S$ cut from the plane $3x + 2y + z = 6$ by the first octant, oriented counterclockwise when viewed from above. Use Stokes’ theorem to calculate the circulation of the vector field $\mathbf{F} = \langle y, x, 0 \rangle$ around $C$.

**Solution.** Stokes’ Theorem says that $\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$. So calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z \end{vmatrix} = \left\langle \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z} x, - \left( \frac{\partial}{\partial x} 0 - \frac{\partial}{\partial z} y \right), \frac{\partial}{\partial x} xz - \frac{\partial}{\partial y} y \right\rangle = \langle -x, -z, -y \rangle.$$

Also, $S$ is given by $z = f(x, y) = 6 - 3x - 2y$, so

$$\mathbf{n} \, d\sigma = + \langle -f_x, -f_y, 1 \rangle \, dA = \langle 3, 2, 1 \rangle \, dA.$$

Let $R$ be the shadow of $S$, i.e., the triangle cut off in the first quadrant of the $xy$-plane by $3x + 2y = 6$. Then

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_R \langle -x, 0, -1 \rangle \cdot \langle 3, 2, 1 \rangle \, dA = \int_R (-3x + 6 - 3x - 2y - 1) \, dA$$

$$= \int_0^2 \int_0^{3-\frac{x}{2}} (5 - 6x - 2y) \, dy \, dx$$

$$= \int_0^2 \left( 5 \left( 3 - \frac{3}{2}x \right) - 6x \left( 3 - \frac{3}{2}x \right) - \left( 3 - \frac{3}{2}x \right)^2 \right) \, dx$$

$$= \int_0^2 \left( 6 - \frac{33}{2}x + \frac{27}{4}x^2 \right) \, dx = \left( 6x - \frac{33}{4}x^2 + \frac{9}{4}x^3 \right) \bigg|^1 = 12 - 33 + 18 = -3.$$

(14) 5. Let $Q$ be the solid inside the cylinder $x^2 + y^2 = 4$ and between the planes $z = 0$ and $z = 3$. Let $S$ be the boundary of $Q$ with outward pointing normal $\mathbf{n}$ and $\mathbf{F} = \langle x + y^2, y + x^2, z + xy \rangle$.

State the Divergence Theorem and use it to evaluate the flux integral $\int_S \mathbf{F} \cdot \mathbf{n} \, dS$.

**Solution.** Gauss’s Divergence says that if $Q$ is a solid whose closed boundary $S$ has outward pointing normal $\mathbf{n}$, then

$$\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_Q \nabla \cdot \mathbf{F} \, dV.$$

Calculate

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x + y^2) + \frac{\partial}{\partial y} (y + x^2) + \frac{\partial}{\partial z} (z + xy) = 1 + 1 + 1 = 3.$$

Hence

$$\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_Q 3 \, dV = 3 \iiint_Q dV,$$

and the latter integral is just the volume of a right circular cylinder of radius 2 and height 3, so that

$$\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = 3 \iiint_Q dV = 3 \pi 2^2 \cdot 3 = 36 \pi.$$

(Or you could work it out the long way with cylindrical coordinates:)

$$\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_Q 3 \, dV = 3 \int_0^{2\pi} \int_0^2 \int_0^3 dz \, dr \, d\theta = 3 \int_0^{2\pi} d\theta \int_0^3 dz \int_0^2 r \, dr = 3 \cdot 2\pi \cdot 3 \cdot 2 = 36 \pi.$$