Exam 4 **Math 208** **Fall 2009**

Name:____________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* Clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(24) **1.** Let a surface S be given by $z = 2x + y^2$ where $(x, y) \in R = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 4\}$.
(a) Find formulas for vector and scalar differential surface area $\mathbf{n}\,d\sigma$ and $d\sigma$ in terms of dA, differential surface area in the xy-plane, where $\mathbf{n}$ is the downward oriented normal.
SOLUTION. The surface is the graph of $f(x, y) = 2x + y^2$, so we have

$$\mathbf{n}\,d\sigma = \pm\langle -f_x, -f_y, 1\rangle\, dA = -\langle -2, -2y, 1\rangle\, dA = \langle 2, 2y, -1\rangle\, dA,$$

where we choose the minus sign for the correct orientation of $\mathbf{n}$. Therefore

$$d\sigma = |\mathbf{n}\,d\sigma| = |\mathbf{n}|\,d\sigma = \sqrt{4 + 4y^2 + 1}dA = \sqrt{5 + 4y^2}dA.$$

(b) Express $\iint_S f(x, y, z)\, d\sigma$ as an iterated integral in x and y where $f(x, y, z) = \sin(xy^2z)$. Do *not* work the integral out.
SOLUTION.

$$\iint_S f(x, y, z)\, d\sigma = \iint_R f\left(x, y, 2x + y^2\right)\sqrt{5 + 4y^2}dA = \int_0^3\int_0^4 f\left(x, y, 2x + y^2\right)\sqrt{5 + 4y^2}dy\, dx$$

(c)Calculate $\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma$, where $\mathbf{F} = \langle 2z, 0, 4y^2\rangle$.
SOLUTION.

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma &= \iint_R \left\langle 2z, 0, 4y^2\right\rangle \langle 2, 2y, -1\rangle\, dA \\ &= \int_0^3\int_0^4 \left(4z - 4y^2\right) dy\, dx \\ &= \int_0^3\int_0^4 \left(4\left(2x + y^2\right) - 4y^2\right) dy\, dx \\ &= \int_0^3\int_0^4 8x dy\, dx \\ &= \frac{8x^2}{2}\Big|_0^3\, y\Big|_0^4 = 36\cdot 4 = 144.\end{aligned}$$

(16) **2.** Apply the flow form of Green's Theorem to compute the area of the ellipse $x^2/4+y^2/9=1$ by using $\mathbf{F}=x\mathbf{j}$. (It may help to recall that this ellipse can be parametrized by $x=2\cos t$, $y=3\sin t$.)

SOLUTION. The flow form of Green's Theorem is, with $\mathbf{T}\cdot ds=d\mathbf{r}=\langle dx,dy\rangle$,

$$\int_C \mathbf{F}\cdot\mathbf{T}ds=\int_C M\,dx+N\,dy=\iint_R(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA=\iint_R(N_x-M_y)\,dA,$$

and in our case $M=0$ and $N=x$, so that $N_x-M_y=1$ and the area of R is

$$\iint_R 1\,dA=\iint_R(N_x-M_y)\,dA=\int_C\mathbf{F}\cdot d\mathbf{r}=\int_C x\,dy=\int_0^{2\pi}2\cos t\,3\cos t\,dt,$$

since the parametrization of C in the hint implies that $dy=3\cos t\,dt$. Thus then area of R is

$$6\int_0^{2\pi}\cos^2 t\,dt=6\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt=6\left(\frac{1}{2}t-\frac{\sin 2t}{4}\right)|_0^{2\pi}=6\left(\frac{2\pi}{2}-0\right)=6\pi.$$

(26) **3.** Let $\mathbf{F}=\langle yz,xz,xy\rangle$.

(a) Show that $\mathbf{F}$ is conservative without actually finding a potential function for $\mathbf{F}$.

SOLUTION. Calculate the curl of $\mathbf{F}$:

$$\begin{aligned}\nabla\times\mathbf{F}&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ yz&xz&xy\end{vmatrix}=\left\langle\frac{\partial}{\partial y}xy-\frac{\partial}{\partial z}xz,-\left(\frac{\partial}{\partial x}xy-\frac{\partial}{\partial z}yz\right),\frac{\partial}{\partial x}xz-\frac{\partial}{\partial y}yz\right\rangle\\ &=\langle x-x,-(y-y),x-x\rangle=\langle 0,0,0\rangle.\end{aligned}$$

Since the curl is zero in a simply connected region (3D space), $\mathbf{F}$ is a conservative vector field.

(b) Find a potential function $f(x,y,z)$ for $\mathbf{F}$.

SOLUTION.

Say $\mathbf{F}=\langle f_x,f_y,f_z\rangle$, so that $f_x=yz$, $f_y=xz$ and $f_z=xy$. Then

$$\begin{aligned}f&=\int f_x dx=xyz+C(y,z),\\ f_y&=xz+C_y(y,z)=xz,\\ C(y,z)&=\int C_y dy=\int 0dy=D(z).\end{aligned}$$

This implies that $f=xyz+D(z)$, so that

$$f_z=xy+D'(z)=xy,$$

so that $D'(z)=0$ and we can take $D=0$. Hence a potential function for $\mathbf{F}$ is

$$f(x,y,z)=xyz.$$

(c) Find the value of the line integral $\int_C yz\,dx+xz\,dy+xy\,dz$, where C is the curve given by position vector $\mathbf{r}=\langle t,\cos t,e^t\rangle$, $0\le t\le\pi$.

SOLUTION. This line integral is just $\int_C\mathbf{F}\cdot d\mathbf{r}$, with $\mathbf{F}=\langle yz,xz,xy\rangle$ and $d\mathbf{r}=\langle dx,dy,dz\rangle$. Therefore, we use the potential function of (b) to obtain

$$\begin{aligned}\int_C yz\,dx+xz\,dy+xy\,dz&=\int_C\mathbf{F}\cdot d\mathbf{r}=f(\pi,\cos\pi,e^\pi)-f\left(0,\cos 0,e^0\right)\\ &=\pi(-1)e^\pi-0\cdot 1e^0=-\pi e^\pi.\end{aligned}$$

(20) **4.** Let C be the boundary of the triangle S cut from the plane $3x + 2y + z = 6$ by the first octant, oriented counterclockwise when viewed from above. Use Stokes' theorem to calculate the circulation of the vector field $\mathbf{F} = \langle y, xz, 0\rangle$ around C.

SOLUTION. Stokes' Theorem says that $\int_C \mathbf{F}\cdot\mathbf{T}ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma$. So calculate

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & 0 \end{vmatrix} = \left\langle \frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}xz, -\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}y\right), \frac{\partial}{\partial x}xz - \frac{\partial}{\partial y}y \right\rangle = \langle -x, 0, z-1\rangle.$$

Also, S is given by $z = f(x,y) = 6 - 3x - 2y$, so

$$\mathbf{n}\,d\sigma = +\langle -f_x, -f_y, 1\rangle\,dA = \langle 3,2,1\rangle\,dA.$$

Let R be the shadow of S, i.e., the triangle cut off in the first quadrant of the xy-plane by $3x+2y=6$. Then

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}ds &= \iint_R \langle -x,0,z-1\rangle\cdot\langle 3,2,1\rangle\,dA = \iint_R (-3x+6-3x-2y-1)\,dA \\ &= \int_0^2\int_0^{3-\frac{3}{2}x} (5-6x-2y)\,dy\,dx \\ &= \int_0^2 \left(5\left(3-\frac{3}{2}x\right) - 6x\left(3-\frac{3}{2}x\right) - \left(3-\frac{3}{2}x\right)^2\right)dx \\ &= \int_0^2 \left(6 - \frac{33}{2}x + \frac{27}{4}x^2\right)dx = \left(6x - \frac{33}{4}x^2 + \frac{9}{4}x^3\right)\Big|_0^2 = 12 - 33 + 18 = -3.\end{aligned}$$

(14) **5.** Let Q be the solid inside the cylinder $x^2+y^2=4$ and between the planes $z=0$ and $z=3$. Let S be the boundary of Q with outward pointing normal $\mathbf{n}$ and $\mathbf{F} = \langle x+y^2, y+x^2, z+xy\rangle$. State the Divergence Theorem and use it to evaluate the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

SOLUTION. Gauss's Divergence says that if Q is a solid whose closed boundary S has outward pointing normal $\mathbf{n}$, then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_Q \nabla\cdot\mathbf{F}\,dV.$$

Calculate

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}\left(x+y^2\right) + \frac{\partial}{\partial y}\left(y+x^2\right) + \frac{\partial}{\partial z}(z+xy) = 1+1+1 = 3.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_Q 3\,dV = 3\iiint_Q dV,$$

and the latter integral is just the volume of a right circular cylinder of radius 2 and height 3, so that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = 3\iiint_Q dV = 3\pi 2^2\cdot 3 = 36\pi.$$

(Or you could work it out the long way with cylindrical coordinates:)

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_Q 3\,dV = 3\int_0^{2\pi}\int_0^2\int_0^3 dz\,r\,dr\,d\theta = 3\int_0^{2\pi}d\theta\int_0^3 dz\int_0^2 r\,dr = 3\cdot 2\pi\cdot 3\cdot 2 = 36\pi.$$