Sample Exam 3 Key **Math 208** **Fall 2009**

Name:______________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(14) **1.** Evaluate the integral $I = \int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz\, dy\, dx$.

SOLUTION. (Exercise 13.5.10) We have, with substitution $u = (3-3x)$, $du = -3dx$, $dx/2 = -du/6$, $u(0) = 3$, $u(1) = 0$, (or just observing $\int \frac{(3-3x)^2}{2} dx = \int \frac{9(x-1)^2}{2} dx = 3\frac{(x-1)^3}{2}$)

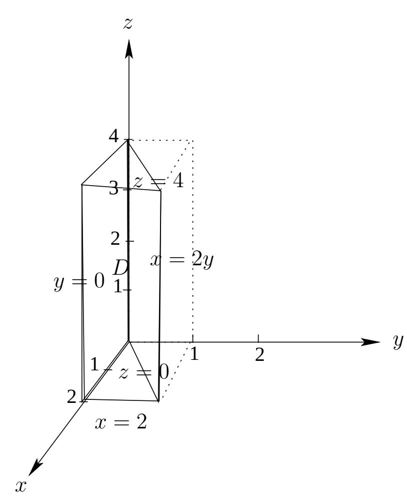
$$\begin{aligned} I &= \int_0^1 \int_0^{3-3x} z\Big|_{z=0}^{3-3x-y} dy\, dx = \int_0^1 \int_0^{3-3x} (3-3x-y)\, dy\, dx \\ &= = \int_0^1 \left((3-3x)\, y - \frac{y^2}{2} \right)\Bigg|_{y=0}^{3-3x} dx = \int_0^1 \left((3-3x)^2 - \frac{(3-3x)^2}{2} \right) dx \\ &= \int_0^1 \frac{(3-3x)^2}{2} dx = -\int_3^0 \frac{u^2}{6} du = \frac{u^3}{18}\Big|_0^3 = \frac{3}{2}. \end{aligned}$$

(18) **2.** Sketch the region D over which the iterated integral I below is calculated. Then express the integral in the order $dy\, dz\, dx$ and write a formula for the average value of $f(x, y, z)$ over D in terms of iterated integrals. Do NOT evaluate any integrals.

$$I = \int_0^4 \int_0^1 \int_{2y}^2 f(x, y, z)\, dx\, dy\, dz.$$

SOLUTION. (Exercise 13.5.41) Region Q is sketched to the right. From it we see that

$$I = \int_0^2 \int_0^4 \int_0^{x/2} f(x, y, z)\, dy\, dz\, dx.$$



Thus, we have that the area of the region is

$$A = \int_0^2 \int_0^4 \int_0^{x/2} dy\, dz\, dx$$

and that the average value of $f(x, y, z)$ over Q is

$$\bar{f}_Q = I/A.$$

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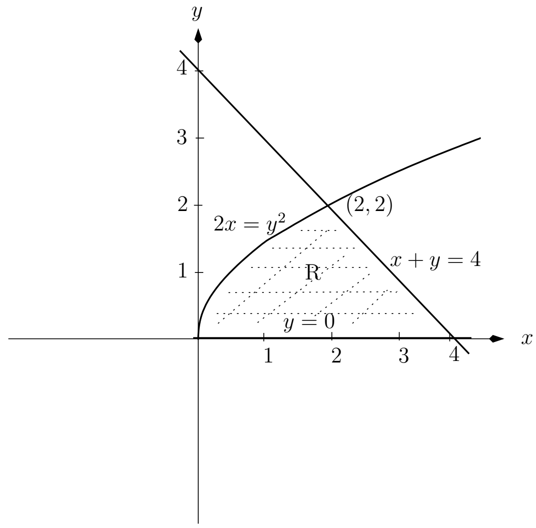
(18) **3.** Sketch the region R in the plane bounded by the curves $y = 0$, $y^2 = 2x$, and $x + y = 4$ and use iterated integrals to write formulas for the area and the first moment of R about the x-axis (assume density $\delta = y^2$) in terms of iterated integrals. Do NOT evaluate any integrals. (Arithmetic check: second and third curves intersect at $(2, 2)$.)
SOLUTION.

(Exercise 13.6.3) The quadratic and straight line intersect where $y^2 = 2(4 - y)$, i.e., where $0 = y^2 + 2y - 8 = (y + 4)(y - 2)$, and at $y = 2$ we have $x = 2^2/2 = 2$. Region R is sketched in the graph to the right. We have that

$$\text{Area(R)} = \iint_R dA = \int_0^2 \int_{y^2/2}^{4-y} dx\, dy$$

(or $\int_0^2 \int_0^{\sqrt{2x}} dy\, dx + \int_2^4 \int_0^{4-x} dy\, dx$) and that the first moment of R about the x-axis, assuming that $\delta(x, y) = y^2$ is given by

$$M_x = \iint_R y\delta dA = \int_0^2 \int_{y^2/2}^{4-y} y^3\, dx\, dy.$$



(20) **4.** A solid D is bounded by the surfaces $z = 1$ and $z = \sqrt{x^2 + y^2}$. Sketch it and express the integral $\iiint_D f(x, y, z)\, dV$ as an iterated integral in both cylindrical and spherical coordinates. Use one of these to express the moment of inertia I_z about the z-axis of a solid occupying D with density function $\delta = z$ as an iterated integral. Do NOT evaluate it.
SOLUTION.

(Exercise 13.7.77) The region is sketched in the figure to the right. The plane $z = 1$ and cone $z = \sqrt{x^2 + y^2}$ intersect in the circle $1 = x^2 + y^2$, which is the shadow R of the region D in the xy-plane. Also, the cone makes an angle of $\pi/4$ with the vertical and the plane $z = 1$ gives $\rho \cos \phi = 1$, i.e., $\rho = 1/\cos \phi = \sec \phi$. Hence, the integral in spherical coordinates is

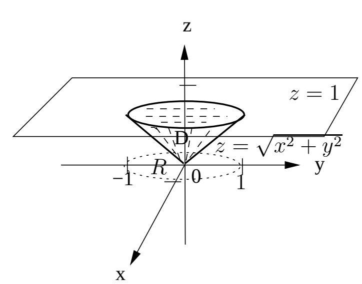
$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta.$$

In cylindrical coordinates the integral is

$$\int_0^{2\pi} \int_0^1 \int_r^1 f(r \cos \theta, r \sin \theta, z)\, dz\, r\, dr\, d\theta.$$

Also in cylindrical coordinates

$$I_z = \iiint_D (x^2 + y^2)\, \delta\, dV = \int_0^{2\pi} \int_0^1 \int_r^1 r^3 z\, dz\, dr\, d\theta.$$



(16) **5.** Evaluate the line integral $\int_C 1\,ds$ where C has position vector $\mathbf{r}(t) = \left\langle \cos t, \sin t, \frac{2}{3}t^{3/2} \right\rangle$, $0 \le t \le 1$.

SOLUTION. (Exercise 14.1.30) The position vector $\mathbf{r}(t)$ gives us a parametrization of C, namely

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= \frac{2}{3}t^{3/2}, \end{aligned}$$

from which we obtain differential formulas

$$\begin{aligned} dx &= -\sin t\, dt \\ dy &= \cos t\, dt \\ dz &= \frac{2}{3}\frac{3}{2}t^{1/2} = t^{1/2}dt, \end{aligned}$$

from which it follows that $ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(-\sin t)^2 + (\cos t)^2 + (t^{1/2})^2} = \sqrt{1+t}dt$. (Or use $ds = |d\mathbf{r}| = \left|\frac{d\mathbf{r}}{dt}\right| dt$.) Thus, with substitution $u = 1 + t$, $du = dt$, $u(0) = 1$, $u(1) = 2$ (or just observing $\int \sqrt{1+t}dt = \frac{2}{3}(1+t)^{3/2}$)

$$\int_C 1\,ds = \int_0^1 \sqrt{1+t}dt = \int_1^2 u^{1/2}du = \frac{2}{3}u^{3/2}\Big|_{u=1}^{2} = \frac{2}{3}\left(2\sqrt{2} - 1\right) = \frac{4}{3}\sqrt{2} - \frac{2}{3}.$$

(14) **6.** Let C be the curve $y = x^2$ traversed from $(0, 0)$ to $(1, 1)$, and $\mathbf{F} = \langle x, y \rangle$ a vector field. Express the flow (i.e., work) of $\mathbf{F}$ along C as a line integral and evaluate it.

SOLUTION. (Exercise 14.2.17) We have $\langle M, N \rangle = \mathbf{F}(x, y) = \langle x, y \rangle$ and that the flow of $\mathbf{F}$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy = \int_C x\, dx + y\, dy.$$

Now parametrize C with position vector $\mathbf{r}(t) = \langle t, t^2 \rangle$ (or $\mathbf{r}(t) = \langle x, x^2 \rangle$) and get

$$\begin{aligned} x &= t \\ y &= t^2,\ 0 \le t \le 1 \end{aligned}$$

so that $dx = dt$, $dy = 2t\, dt$, and obtain that

$$W = \int_0^1 \left(t\, dt + t^2 2t\, dt\right) = \int_0^1 \left(t + 2t^3\right) dt = \left(\frac{t^2}{2} + \frac{2}{4}t^4\right)\Big|_{t=0}^{1} = \frac{1}{2} + \frac{1}{2} - 0 = 1.$$