(14) 1. Evaluate the integral \( I = \int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \, dy \, dx. \)

**SOLUTION.** (Exercise 13.5.10) We have, with substitution \( u = (3 - 3x), \ du = -3dx, \ dx/2 = -du/6, \ u(0) = 3, \ u(1) = 0, \) (or just observing \( \int \frac{(3-3x)^2}{2} \, dx = \int \frac{9(x-1)^2}{2} \, dx = 3 \frac{(x-1)^3}{2} \))

\[
I = \int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz \, dy \, dx = \int_0^1 \int_0^{3-3x} (3-3x-y) \, dy \, dx
\]

\[
= \int_0^1 \left( (3-3x) \frac{y^2}{2} \right)_{y=0}^{y=3-3x} \, dx = \int_0^1 \left( (3-3x)^2 - \frac{(3-3x)^2}{2} \right) \, dx
\]

\[
= \int_0^1 \frac{(3-3x)^2}{2} \, dx = -\int_0^3 \frac{u^2}{6} \, du = \frac{u^3}{18} \bigg|_0^3 = \frac{3}{2}.
\]

(18) 2. Sketch the region \( D \) over which the iterated integral \( I \) below is calculated. Then express the integral in the order \( dy \, dz \, dx \) and write a formula for the average value of \( f(x, y, z) \) over \( D \) in terms of iterated integrals. Do NOT evaluate any integrals.

\[
I = \int_0^4 \int_0^1 \int_{2y}^2 f(x, y, z) \, dx \, dy \, dz.
\]

**SOLUTION.** (Exercise 13.5.41) Region \( Q \) is sketched to the right. From it we see that

\[
I = \int_0^2 \int_0^4 \int_{x/2}^4 f(x, y, z) \, dy \, dz \, dx.
\]

Thus, we have that the area of the region is

\[
A = \int_0^2 \int_0^4 \int_{x/2}^4 dy \, dz \, dx
\]

and that the average value of \( f(x, y, z) \) over \( Q \) is

\[
\bar{f}_Q = I/A.
\]
3. Sketch the region $R$ in the plane bounded by the curves $y = 0$, $y^2 = 2x$, and $x + y = 4$ and use iterated integrals to write formulas for the area and the first moment of $R$ about the $x$-axis (assume density $\delta = y^2$) in terms of iterated integrals. Do NOT evaluate any integrals. (Arithmetic check: second and third curves intersect at $(2, 2)$.)

Solution.

(Exercise 13.6.3) The quadratic and straight line intersect where $y^2 = 2(4 - y)$, i.e., where $0 = y^2 + 2y - 8 = (y + 4)(y - 2)$, and at $y = 2$ we have $x = 2^2/2 = 2$. Region $R$ is sketched in the graph to the right. We have that

$$\text{Area}(R) = \int \int _R dA = \int _0 ^2 \int _{y^2/2} ^{4-y} dx \, dy$$

(or $\int _0 ^2 \int _0 ^{\sqrt{2x}} dy \, dx + \int _2 ^4 \int _0 ^{4-x} dy \, dx$) and that the first moment of $R$ about the $x$-axis, assuming that $\delta(x, y) = y^2$ is given by

$$M_x = \int \int _R y \delta dA = \int _0 ^2 \int _{y^2/2} ^{4-y} y^3 \, dx \, dy.$$

4. A solid $D$ is bounded by the surfaces $z = 1$ and $z = \sqrt{x^2 + y^2}$. Sketch it and express the integral $\int \int \int _D f(x, y, z) \, dV$ as an iterated integral in both cylindrical and spherical coordinates. Use one of these to express the moment of inertia $I_z$ about the $z$-axis of a solid occupying $D$ with density function $\delta = z$ as an iterated integral. Do NOT evaluate it.

Solution.

(Exercise 13.7.77) The region is sketched in the figure to the right. The plane $z = 1$ and cone $z = \sqrt{x^2 + y^2}$ intersect in the circle $1 = x^2 + y^2$, which is the shadow $R$ of the region $D$ in the $xy$-plane. Also, the cone makes an angle of $\pi/4$ with the vertical and the plane $z = 1$ gives $\rho \cos \phi = 1$, i.e., $\rho = 1/\cos \phi = \sec \phi$. Hence, the integral in spherical coordinates is

$$I_z = \int \int \int _D (x^2 + y^2) \delta \, dV = \int _0 ^{2\pi} \int _0 ^1 \int _0 ^{\sec \phi} \rho^3 z \, dz \, d\rho \, d\phi \, d\theta.$$

In cylindrical coordinates the integral is

$$\int _0 ^{2\pi} \int _0 ^1 \int _r ^1 f(r \cos \theta, r \sin \theta, z) \, dz \, r \, dr \, d\theta.$$
5. Evaluate the line integral \( \int_C 1 \, ds \) where \( C \) has position vector \( \mathbf{r}(t) = \langle \cos t, \sin t, \frac{2}{3} t^{3/2} \rangle \), \( 0 \leq t \leq 1 \).

**Solution.** (Exercise 14.1.30) The position vector \( \mathbf{r}(t) \) gives us a parametrization of \( C \), namely

\[
\begin{align*}
x &= \cos t \\
y &= \sin t \\
z &= \frac{2}{3} t^{3/2},
\end{align*}
\]

from which we obtain differential formulas

\[
\begin{align*}
dx &= -\sin t \, dt \\
dy &= \cos t \, dt \\
dz &= \frac{2}{3} t^{1/2} = t^{1/2} \, dt,
\end{align*}
\]

from which it follows that

\[
ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(-\sin t)^2 + (\cos t)^2 + (t^{1/2})^2} = \sqrt{1 + t} \, dt.
\]

(Or use \( ds = |d\mathbf{r}| = |\frac{d\mathbf{r}}{dt}| \, dt \).) Thus, with substitution \( u = 1 + t \), \( du = dt \), \( u(0) = 1 \), \( u(1) = 2 \) (or just observing \( \int \sqrt{1 + t} \, dt = \frac{2}{3} (1 + t)^{3/2} \))

\[
\int_C 1 \, ds = \int_0^1 \sqrt{1 + t} \, dt = \int_1^2 u^{1/2} \, du = \left[ \frac{2}{3} u^{3/2} \right]_{u=1}^{u=2} = \frac{2}{3} \left( 2\sqrt{2} - 1 \right) = \frac{4}{3} \sqrt{2} - \frac{2}{3}.
\]

6. Let \( C \) be the curve \( y = x^2 \) traversed from \((0,0)\) to \((1,1)\), and \( \mathbf{F} = \langle x, y \rangle \) a vector field. Express the flow (i.e., work) of \( \mathbf{F} \) along \( C \) as a line integral and evaluate it.

**Solution.** (Exercise 14.2.17) We have \( \langle M, N \rangle = \mathbf{F}(x,y) = \langle x, y \rangle \) and that the flow of \( \mathbf{F} \) along \( C \) is

\[
W = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy = \int_C x \, dx + y \, dy.
\]

Now parametrize \( C \) with position vector \( \mathbf{r}(t) = \langle t, t^2 \rangle \) (or \( \mathbf{r}(t) = \langle x, x^2 \rangle \)) and get

\[
\begin{align*}
x &= t \\
y &= t^2, \quad 0 \leq t \leq 1
\end{align*}
\]

so that \( dx = dt \), \( dy = 2t \, dt \), and obtain that

\[
W = \int_0^1 (t \, dt + t^2 2t \, dt) = \int_0^1 (t + 2t^3) \, dt = \left( \frac{t^2}{2} + \frac{2}{4} t^4 \right) \bigg|_{t=0}^{t=1} = \frac{1}{2} + \frac{1}{2} - 0 = 1.
\]