Exam 3 **Math 208** **Fall 2009**

Name:______________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* Clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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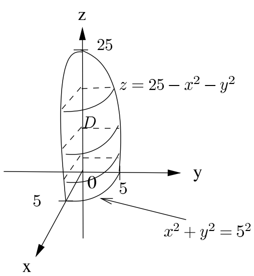
(20) **1.** Let D be the mass solid in the first octant $(x, y, z \geq 0)$ below the surface $z = 25 - x^2 - y^2$ and with density function $\delta(x, y, z) = y$. Sketch this solid. Express the mass and the center of mass of the solid in terms of iterated integrals in cylindrical coordinates. *Do not evaluate the integrals.*

SOLUTION.



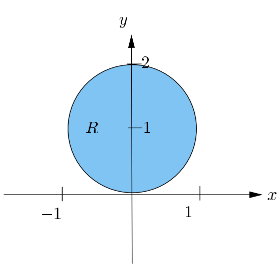
The mass is M and center of mass is $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$, where

$$\begin{aligned}
M &= \iiint_D y \cdot dV = \int_0^{\pi/2}\int_0^5\int_0^{25-r^2} r^2 \sin\theta \, dz\, dr\, d\theta \\
M_{yz} &= \iiint_D xy \cdot dV = \int_0^{\pi/2}\int_0^5\int_0^{25-r^2} r^3 \cos\sin\theta \, dz\, dr\, d\theta \\
M_{xz} &= \iiint_D y^2 \cdot dV = \int_0^{\pi/2}\int_0^5\int_0^{25-r^2} r^3 \sin^2\theta \, dz\, dr\, d\theta \\
M_{xy} &= \iiint_D zy \cdot dV = \int_0^{\pi/2}\int_0^5\int_0^{25-r^2} zr^2 \sin\theta \, dz\, dr\, d\theta.
\end{aligned}$$

(15) **2.** Find the mass of a lamina that occupies the plane region R bounded by the circle $x^2+y^2 = 2y$ and has density function $\delta(x, y) = 1/\sqrt{x^2+y^2}$. Sketch R.

SOLUTION.

The region R is a circle of radius 1 and center $(0, 1)$. Its equation in polar coordinates is $r^2 = 2r\sin\theta$ or just $r = 2\sin\theta$.



The mass of the lamina is given by

$$\begin{aligned}
M &= \iint_R \delta \cdot dV \\
&= \int_0^{\pi}\int_0^{2\sin\theta} \frac{1}{r} r \, dr\, d\theta \\
&= \int_0^{\pi} (2\sin\theta - 0)\, d\theta \\
&= -2\cos\theta|_0^{\pi} \\
&= -2(-1-1) = 4.
\end{aligned}$$

(17) **3.** Evaluate the iterated integral I defined by

$$I = \int_0^1 \int_0^{3-x} \int_0^{3-2y} dz\, dy\, dx.$$

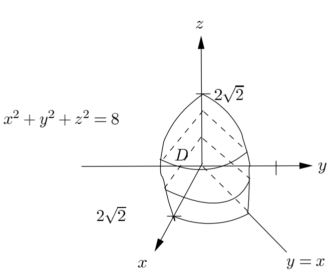
SOLUTION. We have

$$\begin{aligned}
I &= \int_0^1 \int_0^{3-x} z|_{z=0}^{3-3x-2y} dy\, dx \\
&= \int_0^1 \int_3^{3-x} (3-2y)\, dy\, dx \\
&= \int_0^1 \left(3y - y^2\right)|_{y=0}^{3-x} dx \\
&= \int_0^1 \left(3\,(3-x) - (3-x)^2\right) dx \\
&= \int_0^1 \left(9 - 3x - 9 + 6x - x^2\right) dx \\
&= \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)|_{x=0}^1 = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}
\end{aligned}$$

(Anyone who got a negative answer should have rechecked, since this is evidently the volume of a solid – the integrand is 1.)

(18) **4.** Express the integral of $f(x,y,z) == 1/\sqrt{x^2+y^2+z^2}$ over the region bounded by surfaces $x^2+y^2+z^2 = 8$, $y = 0$ and $y = x$, with $x, y \geq 0$, as an iterated integral in spherical coordinates and evaluate it.
SOLUTION.

(Handout #3) The solid is a half-sector of the sphere bounded laterally by the xz-plane and the vertical plane $y = x$, and below by the xy-plane. So θ ranges from 0 to $\pi/4$ over the solid, ϕ ranges from 0 to $\pi/2$, and ρ from 0 to $\sqrt{8} = 2\sqrt{2}$. The sketch of the solid D:



From this we see that the integral is

$$\begin{aligned}
\iiint_D f(x,y,z)\, dV &= \iiint_D \frac{1}{\rho}\, dV \\
&= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{8}} \frac{1}{\rho}\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi/4} \int_0^{\pi/2} \sin\phi\, \frac{\rho^2}{2}|_0^{\sqrt{8}}\, d\rho\, d\phi\, d\theta \\
&= 4\int_0^{\pi/4} -\cos\phi|_0^{\pi/2}\, d\theta \\
&= 4\,(1-0)\,\theta|_0^{\pi/4} \\
&= 4\frac{\pi}{4} = \pi
\end{aligned}$$

(15) **5.** Compute $\int_C \frac{y}{2}\, ds$ along the curve given by $\mathbf{r}(t) = (t^2-1)\,\mathbf{j} + 2t\mathbf{k}$, $0 \le t \le 1$.
SOLUTION.

(Exer. 14.1.23) The parametrization for this curve for $0 \le t \le 1$ is

$$\begin{aligned} x &= 0 \\ dx &= 0 \\ y &= t^2-1 \\ dy &= 2t\,dt \\ z &= 2t \\ dz &= 2dt \\ ds &= \sqrt{dx^2+dy^2+dz^2} = \sqrt{0+4t^2dt^2+4dt^2} \\ &= 2\sqrt{t^2+1}dt. \end{aligned}$$

Therefore, the integral is

$$\int_C y\,ds = \int_0^1 \left(t^2-1\right)\cdot 2\sqrt{t^2+1}dt.$$

This integral requires a trig substitution, so I will give full credit for getting this far. (Actual answer: $\frac{5}{4}\ln\left(\sqrt{2}-1\right) - \frac{\sqrt{2}}{4}$ by doing a trig substitution with right triangle with legs t and 1, hypotenuse $\sqrt{1+t^2}$. Integral was supposed to be $\int_C \frac{z}{2}\, ds$.)

(15) **6.** Find the work done by the force field $\mathbf{F} = \langle -y, x\rangle$ in moving an object around the circle C parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Also calculate the flux of $\mathbf{F}$ across C.
SOLUTION.

(Exer. 14.2.23) For $0 \le t \le 2\pi$ we have

$$\begin{aligned} x &= \cos t \\ dx &= -\sin t\,dt \\ y &= \sin t \\ dy &= \cos t\,dt, \end{aligned}$$

so that the work done is

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C -y\,dx + x\,dy \\ &= \int_0^{2\pi} \left\{(-\sin t)^2\,dt + (\cos t)^2\right\}dt \\ &= \int_0^{2\pi} 1dt = 2\pi. \end{aligned}$$

Also, the flux across the curve is given by

$$\begin{aligned} \int_C \mathbf{F}\cdot\langle dy, -dx\rangle &= \int_C -y\,dy + x\,dx \\ &= \int_0^{2\pi} \{-\sin t\,\cos t\,dt + \cos t\,\sin t\}\,dt \\ &= \textstyle\int_0^{2\pi} 0\,dt \\ &= 0. \end{aligned}$$