(24) 1. Let \( f(x, y) = 8x^2 + 4x^2y + y^2 - 7 \).

(a) Find all derivatives up to the second order.

\[
\begin{array}{|c|c|}
\hline
f_x & f_y \\
\hline
16x + 8xy & 4x^2 + 2y \\
\hline
f_{xx} & f_{yy} \\
\hline
16 + 8y & 2 \\
\hline
f_{xy} & f_{yx} \\
\hline
8x & 8x \\
\hline
\end{array}
\]

(b) Find all critical points of \( f \).

Set \( 0 = f_x = 16x + 8xy \) and \( 0 = 4x^2 + 2y \).

From the second equation \( y = -2x^2 \).

Plug into first and obtain \( 0 = 16x - 16x^3 = 16x(1 - x^2) \), so \( x = 0, -1, 1 \).

For each \( x \) obtain corresponding \( y = 0, -2, -2 \).

Critical points are \((0, 0), (-1, -2), (1, -2)\).

(c) Use the second derivative test to classify the critical points of \( f \).

The discriminant is \( D_f = f_{xx}f_{yy} - (f_{xy})^2 = (16 + 8y)^2 - (8x)^2 = 32 + 16y - 64x^2 \).

\( D_f(0, 0) = 32 \), so we have a local max/min. Since \( f_{yy}(0, 0) = 2 > 0 \), \( f \) has a local minimum at \((0, 0)\).

\( D_f(-1, -2) = 0 - 64 \cdot 1 = -64 < 0 \), so \( f \) has a saddle point at \((-1, -2)\).

\( D_f(1, -2) = 0 - 64 \cdot 1 = -64 < 0 \), so \( f \) has a saddle point at \((1, -2)\).
2. Let \( f(x, y) = xy \).
(a) Find the extrema of \( f \) subject to the constraint \( x^2 + 2y^2 = 1 \) by the method of Lagrange multipliers.

The Lagrange equations are
\[
g(x, y) = x^2 + 2y^2 - 1 = 0 \quad \text{and} \quad \langle y, x \rangle = \langle f_x, f_y \rangle = \nabla f = \lambda \nabla g = \langle \lambda g_x, \lambda g_y \rangle = \langle \lambda 2x, \lambda 4y \rangle.
\]
So the system is
\[
y = 2\lambda x \\
x = 4\lambda y \\
x^2 + 2y^2 = 1.
\]
If \( x = 0 \), first equation implies \( y = 0 \), which contradicts the third equation.
Likewise, if \( y = 0 \), second equation implies \( x = 0 \), which again contradicts the third.
So neither is zero. Solve for \( \lambda \) in the first and second and obtain that \( \frac{y}{2x} = \lambda = \frac{x}{4y} \), so \( 4y^2 = 2x^2 \) and \( x^2 = 2y^2 \). Plug this into the third equation and get \( 2y^2 + 2y^2 = 4y^2 = 1 \), so \( y = \pm \frac{1}{2} \). For each such \( y \), \( x^2 = 2 \left( \frac{1}{4} \right) = \frac{1}{2} \), so \( x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \).

This gives 4 critical points, \( \left( \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{2} \right) \). Now evaluate:

\[
f \left( \frac{1}{\sqrt{2}}, \frac{1}{2} \right) = \frac{1}{2\sqrt{2}}, \text{ a MAX value.}
\]
\[
f \left( \frac{1}{\sqrt{2}}, -\frac{1}{2} \right) = -\frac{1}{2\sqrt{2}}, \text{ a MIN value.}
\]
\[
f \left( -\frac{1}{\sqrt{2}}, \frac{1}{2} \right) = -\frac{1}{2\sqrt{2}}, \text{ a MIN value.}
\]
\[
f \left( -\frac{1}{\sqrt{2}}, -\frac{1}{2} \right) = \frac{1}{2\sqrt{2}}, \text{ a MAX value.}
\]

(b) What additional point(s) should you check to find the absolute extrema of \( f \) over the region \( x^2 + 2y^2 \leq 1 \)?

Use EVT. We checked boundary points of this closed bounded region in (a). We should also check critical points of \( f(x, y) \) that are interior to the region. In this case, \( 0 = f_x = y \) and \( 0 = f_y = x \) gives a single critical point \( (0, 0) \) in the interior.

(18) 3. Express the volume of the solid bounded above by the paraboloid \( z = x^2 + y^2 \) and below by the rectangle \( R : 0 \leq x \leq 1, 0 \leq y \leq 1 \) as a double integral and evaluate this integral.

The volume is given by the double integral
\[
\iint_R (x^2 + y^2) \, dA = \int_0^1 \int_0^1 (x^2 + y^2) \, dx \, dy
\]
\[
= \int_0^1 \left[ \frac{x^3}{3} + y^2 x \right]_{x=0}^{1} \, dy
\]
\[
= \int_0^1 \left( \frac{1}{3} + y^2 \right) \, dy
\]
\[
= \left( \frac{y^3}{3} + \frac{y^3}{3} \right)_{y=0}^{1} = \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{2}{3}
\]
4. Evaluate the integral
\[ \int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx \]
by interchanging the order of integration. Clearly sketch the region of integration.

The region \( R \) is a triangle with vertices \((0, 0)\), \((0, \pi)\) and \((\pi, \pi)\). The integral is

\[ \int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx = \int_0^\pi \int_y^\pi \frac{\sin y}{y} \, dx \, dy \]
\[ = \int_0^\pi \int_0^y \sin y \, dx \, dy \]
\[ = \int_0^\pi \sin y \left[ x \right]_0^y \, dy \]
\[ = \int_0^\pi \sin y \, dy \]
\[ = \cos y \bigg|_{y=0}^{y=\pi} \]
\[ = \cos \pi - \cos 0 \]
\[ = -1 - 1 = -2. \]

5. Convert the iterated integral
\[ \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-\frac{1}{x^2}}} (x^2 + y^2) \, dy \, dx \]
to polar coordinates and evaluate. Sketch the region of integration for this problem. What is the average value of \( f(x, y) = x^2 + y^2 \) over this region?

The region \( R \) is the portion in the first quadrant of the disk with radius 1 and center at the origin.

The integral is

\[ \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx = \int_0^{\pi/2} \int_0^1 (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \, r \, dr \, d\theta \]
\[ = \int_0^{\pi/2} \frac{1}{2} \left[ \frac{r^4}{4} \right]_0^1 \, d\theta \]
\[ = \int_0^{\pi/2} \frac{1}{4} \, d\theta \]
\[ = \frac{\theta}{4} \bigg|_{\theta=0}^{\theta=\pi/2} \]
\[ = \frac{\pi}{8}. \]

The average value of \( f \) on \( R \) is \( f_R = \frac{1}{\text{Area}(R)} \iint_R f(x, y) \, dA \). In this case the area of a quarter circle of radius 1 is \( \pi/4 \) and we have calculated \( \iint_R f(x, y) \, dA = \pi/8 \), so \( f_R = \frac{1}{\pi/4} \cdot \frac{\pi}{8} = \frac{1}{2}. \)