Exam 2 **Math 208** **Fall 2009**

Name:________________________ **Score:**__________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* Clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Make all obvious simplifications, e.g., 0 rather than $\sin\pi$. Point values of problems are given in parentheses. Notes or text in *any* form not allowed. The only electronic equipment allowed is a calculator.

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(24) **1.** Let $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.
(a) Find all critical points of f.

SOLUTION. (Exer. 12.7.19) Set first derivatives equal to zero and solve for x, y:

$$\begin{aligned} 0 &= f_x = 12x - 6x^2 + 6y \\ 0 &= f_y = 6y + 6x. \end{aligned}$$

Thus $y = -x$ from the second equation. Substitute into the first and obtain

$$0 = 12x - 6x^2 - 6x = 6x - 6x^2,$$

so that $x(1-x) = 0$ and $x = 0$ or $x = 1$.
If $x = 0$, $y = 0$.
If $x = 1$, then $y = -1$.
So the critical points are $(0,0)$ and $(1,-1)$.
(b) Use the second derivative test to classify the critical points of f.
SOLUTION. Find the second derivatives and discriminant:

$$\begin{aligned} f_{xx} &= 12 - 12x \\ f_{xy} &= 6 = f_{yx} \\ f_{yy} &= 6 \\ D_f &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12(1-x) & 6 \\ 6 & 6 \end{vmatrix} = 72(1-x) - 36. \end{aligned}$$

Now check
$D_f(0,0) = 72 - 36 = 36 > 0$,so f has a local max or min. Since $f_{yy}(0,0) = 6 > 0$, f has a local min at $(0,0)$.
$D_f(1,-1) = 0 - 36 < 0$, so that f has a saddle point at $(1,-1)$.

(c) Sketch the region R bounded by the curves $y = 1$ and $y = x^2 - 1$. To find the extrema of $f(x,y)$ in the region, what points would you check? (You do NOT have to calculate any critical points along curves or function values; just indicate what you would do.)
SOLUTION. Sketch should be a pararola opening upward with top horizontal line at $y = 1$.
First check the corners of the region. The curves intersect at $x = \pm\sqrt{2}$, so the corners of the region are $\left(\pm\sqrt{2}, 1\right)$.
Next check any interior critical points, which in this case includes only $(0,0)$.
Finally, find critical points along the curves $f(x, y(x))$ on the interval $[-2,2]$ with $y(x) = 1$ and $y(x) = x^2 - 1$ and check the points resulting points $(x, y(x))$.

(18) **2.** Find the average height of the surface $z = \sin(x+y)$ over the rectangle R given by $0 \le x \le \pi$, $0 \le y \le \pi/2$.

SOLUTION. (Exer 13.3.15(b)) First note the area of R is $\pi^2/2$. Next calculate

$$\begin{aligned}\iint_R \sin(x+y)\,dA &= \int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\,dx\,dy \\ &= \int_0^{\pi/2} \left(-\cos(x+y)\,|_{x=0}^{\pi}\right) dy \\ &= \int_0^{\pi/2} (\cos(y) - \cos(\pi+y))\,dy \\ &= (\sin(y) - \sin(\pi+y))\,|_{y=0}^{\pi/2} \\ &= 1-(-1)-0 = 2.\end{aligned}$$

Therefore the average value of the height is

$$\frac{1}{\text{area of R}}\iint_R \sin(x+y)\,dA = \frac{1}{\pi^2/2}2 = \frac{4}{\pi^2}.$$

(20) **3.** Use Lagrange multipliers to find the extrema of $f(x,y) = x^2 + y^2$ on the curve $xy^2 = 54$.

SOLUTION. (Exer. 12.8.5) Solve the equations $g(x,y) = xy^2 - 54 = 0$ and $\nabla f = \langle f_x, f_y\rangle = \lambda\nabla g = \langle \lambda g_x, \lambda g_y\rangle$, that is

$$\begin{aligned}2x &= \lambda y^2 \\ 2y &= \lambda 2xy \\ xy^2 &= 54.\end{aligned}$$

From the third equation we see that neither x nor y may be zero. So eliminate λ by solving the first two equations for λ and equating to obtain

$$\begin{aligned}\frac{2x}{y^2} &= \frac{2y}{2xy} \\ 2x^2y &= y^3\end{aligned}$$

so that

$$2x^2 = y^2.$$

Hence $54 = xy^2 = 2x^3$, so that $x^3 = 27$ and $x = 3$. It follows that $y^2 = 2x^2 = 18$, so that $y = \pm 3\sqrt{2}$. Check that $f\left(3, \pm 3\sqrt{2}\right) = 9 + 18 = 27$, so this must be the minimum value of f on this curve, since f is nonnegative and unbounded.

(20) **4.** Evaluate the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3} dy\, dx$$

by interchanging the order of integration. Clearly sketch the region R of integration.
SOLUTION. Sketch shows that the region R is above curve $y = \sqrt{x}$ and below the horizontal line $y = 1$, and bounded on the left by the y-axis:

[Picture goes here]

Thus,

$$\begin{aligned} \int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3} dy\, dx &= \iint_R \frac{3}{4+y^3} dA = \int_0^1 \int_0^{y^2} \frac{3}{4+y^3} dx\, dy \\ &= \int_0^1 \frac{3}{4+y^3} \int_0^{y^2} dx\, dy = \int_0^1 \frac{3\left(y^2-0\right)}{4+y^3} dy \\ &= \int_4^5 \frac{du}{u} = \ln 5 - \ln 4 = \ln 1.25, \end{aligned}$$

where the last integral comes from the substitution $u = 4 + y^3$, $du = 3y^2 dy$, $u(0) = 4$, $u(1) = 5$.

(18) **5.** Convert the iterated integral $\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x^2+y^2\right) dx\, dy$ to polar coordinates and evaluate. Sketch the region R of integration for this problem.
SOLUTION. (Exer. 13.4.3) Sketch below shows that region R is the quarter circle of $x^2 + y^2 = 1$ in the first quadrant:

[Picture goes here]

Thus,

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \left(x^2+y^2\right) dx\, dy = \iint_R \left(x^2+y^2\right) dA = \int_0^{\pi/2} \int_0^1 r^2 r\, dr\, d\theta = \int_0^{\pi/2} d\theta \int_0^1 r^3 dr = \frac{\pi}{2}\frac{1}{4} = \frac{\pi}{8}.$$