1. Given points \( P = (0, 2, -1) \) \( Q = (-1, 0, 0) \), \( a = \overrightarrow{PQ} \) and \( n = 3i - 2j - k \). (Exer. 10.5.21)

(a) Find an equation for the plane through \( P \) and perpendicular (normal) to \( n \).

**Solution.** (5) This is given by

\[
0 = n \cdot \langle x - 0, y - 2, z - -1 \rangle
0 = \langle 3, -2, -1 \rangle \cdot \langle x - 0, y - 2, z - -1 \rangle
0 = 3x - 2(y - 2) - 1(z + 1)
0 = 3x - 2y - z + 3.
\]

(b) Find a vector orthogonal to both \( a \) and \( n \).

**Solution.** (4) This is given by the cross-product

\[
v = a \times n = \begin{vmatrix}
i & j & k \\
-1 & -2 & 1 \\
3 & -2 & -1 \\
\end{vmatrix} = ((-2)(-1) - 1(-2), -((1)(-1) - 1 \cdot 3), ((-1)(-2) - (-2)(3)))
= \langle 4, 2, 8 \rangle
\]

(c) Find parametric equations for a line through the point \( P \) and parallel to \( a \).

**Solution.** (5) These are

\[
x = 0 - 1t = -t
y = 2 - 2t
z = -1 + t
\]

where \( t \) is any real number.

(d) Show that the line of (c) is contained in the plane of (a).

**Solution.** (4) Plug coordinates of a general point (or any two particular points) into the plane equation:

\[
3(-t) - 2(2 - 2t) - z(-1 + t) + 3 = -3t + 4t - t - 4 + 1 + 3 = 0,
\]

so the equation of the plane is satisfied.

2. Let \( f(x, y) = \sqrt{9 - x^2 - y^2} \). (Exer 12.1.8)

(a) Find the domain and range of \( f \). Are these sets open, closed, or bounded?

**Solution.** (7) Need \( 9 - x^2 - y^2 \geq 0 \), i.e., \( x^2 + y^2 \leq 9 \). So domain is

\[
D = \{(x, y) \mid x^2 + y^2 \leq 9\},
\]

which is a closed bounded set. The set of possible values goes from 0 to \( \sqrt{9} = 3 \), that is, the closed bounded interval \([0, 3]\).

(b) Describe the contour curves of \( f \) and plot three of them.

**Solution.** (5) From \( z = \sqrt{9 - x^2 - y^2} \), get \( x^2 + y^2 = 9 - z^2 \), so these curves are circles of radius \( \sqrt{9 - z^2} \). One could draw circles of radius 3, \( \sqrt{3} \) and 0 (just a point) in the \( xy \)-plane and label them \( z = 0, 2, 3 \), resp.
2. (continued) (c) What is \( \lim_{(x,y) \to (0,0)} f (x, y) ? \)

**Solution.** Since \( f \) is certainly continuous there (evaluation works on this algebraic expression), so the answer is
\[
\lim_{(x,y) \to (0,0)} f (x, y) = f (0, 0) = \sqrt{9 - 0} = 3.
\]

(17) 3. Find the directional derivative of \( f(x, y) = - (x^2 + y^2) / 4 \) at \( P_0 = (-3, 4) \) in the direction from \( P_0 \) toward \((0, 1)\). In what direction from \( P_0 \) is the rate of increase of \( f \) the greatest? The least? Equal to zero? (Exer. 1, Gradient applications handout)

**Solution.** We calculate
\[
\nabla f (x, y) = - \frac{1}{4} (2x, 2y) = - \frac{1}{2} (x, y),
\]
so that
\[
\nabla f (P_0) = - \frac{1}{2} (-3, 4) = \frac{1}{2} (3, -4).
\]

A vector in the specified direction is
\[
(0 - -3, 1 - 4) = (3, -3) = 3 (1, -1),
\]
so a unit vector in that direction is \( u = \frac{1}{\sqrt{2}} (1, -1) \) and therefore the directional derivative in that direction is
\[
u \cdot \nabla f (-3, 4) = \frac{1}{\sqrt{2}} (1, -1) \cdot \frac{1}{2} (3, -4) = \frac{1}{2\sqrt{2}} (3 + 4) = \frac{7}{4\sqrt{2}} = \frac{7}{4} \sqrt{2} \approx 2.47.
\]

The direction of greatest increase of \( f \) is just \( \nabla f (P_0) = \frac{1}{2} (3, -4) \).

The direction of least rate of increase of \( f \) is \( - \nabla f (P_0) = \frac{1}{2} (-3, 4) \).

The direction of zero rate of increase of \( f \) is any vector orthogonal to \( \nabla f (P_0) \), so \( \frac{1}{2} (4, 3) \) or its negative works for this.

(17) 4. Let \( f (x, y, z) = y^2 - x^2 - z \). What quadric is the level surface \( f (x, y, z) = 0 \)? Find equations for the normal line and tangent plane to this surface at the point \((1, 2, 3)\).

**Solution.** First, the surface \( z = y^2 - x^2 \) is a hyperbolic paraboloid.

Next, to find a normal, calculate \( \nabla f = (-2x, 2y, -1) \), so that a normal vector is
\[
\nabla f (1, 2, 3) = (-2 \cdot 1, 2 \cdot 2, -1) = (-2, 4, -1)
\]
and the equation of the tangent plane is
\[
0 = \nabla f (1, 2, 3) \cdot (x - 1, y - 2, z - 3) = -2 (x - 1) + 4 (y - 2) - 1 (z - 3),
\]
that is,
\[
-2x + 4y - z - 3 = 0.
\]

The normal line is given parametrically by
\[
x = 1 - 2t \\
y = 2 + 4t \\
z = 3 - t
\]
(15) 5. Given a function \( z(s, t) = f(x(s, t), y(s, t)) \), write a chain rule formula for \( z_t \). Use this formula and the additional information that \( x = 3st, y = 8s/t^2 \), \( f_x(-6, 2) = 5 \) and \( f_y(-6, 2) = -3 \) to find \( z_t(1, -2) \). (Exer. 4, Chain rule handout)

Solution. The dependent variable is \( z \), intermediate variables are \( x, y \) and independent variables are \( s, t \). Thus the chain rule formula for \( z_t = \frac{\partial z}{\partial t} \) is

\[
\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.
\]

Now at \( s = 1 \) and \( t = -2 \) we have \( x = 3 \cdot 1 \cdot (-2) = -6 \), and \( y = 8/(-2)^2 = 2 \), so that

\[
(x(1, -2), y(1, -2)) = (-6, 2).
\]

Furthermore,

\[
\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(3st) = 3s \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left(\frac{8s}{t^2}\right) = -\frac{8s}{t^3} = -\frac{16s}{6}.
\]

Therefore

\[
z_t(1, -2) = f_x(-6, 2) 3 \cdot 1 + f_y(-6, 2) \frac{-16 \cdot 1}{(-2)^4} = 5 \cdot 3 - 3 \frac{-16}{8} = 15 - 6 = 9.
\]

(18) 6. Let \( f(x, y) = \sqrt{y^2 - x^2} \). (Exer. 5, Total differentials handout)

(a) Compute the total differential of this function.

Solution. (6) We have the total differential

\[
dz = f_x \, dx + f_y \, dy = \frac{1}{2} \frac{-2x}{\sqrt{y^2 - x^2}} \, dx + \frac{1}{2} \frac{2y}{\sqrt{y^2 - x^2}} \, dy = \frac{-x \, dx + y \, dy}{\sqrt{y^2 - x^2}}.
\]

(b) Given that your error in measuring \( x \) could be as large as 0.4 and the error in measuring \( y \) could be as large as 0.2, use differentials to estimate the possible error in evaluating \( f(x, y) \) as \( f(12, 13) \).

Solution. (6) Evaluate the differential at \( x = 12, y = 13 \), let \( dx \) and \( dy \) be the error in \( x \) and \( y \), and obtain that the error in \( z = f(x, y) \), \( \Delta z \), satisfies

\[
\Delta z \approx dz = \frac{-12 \, dx + 13 \, dy}{\sqrt{13^2 - 12^2}}.
\]

Thus

\[
|\Delta z| \approx \left| \frac{-12 \, dx + 13 \, dy}{5} \right| \leq \frac{1}{5} \{12 \, |dx| + 13 \, |dy|\} \leq \frac{1}{5} \{12 \cdot 0.4 + 13 \cdot 0.2\} = \frac{37}{25} = 1.48.
\]

(c) Compute the linearization \( L(x, y) \) of \( f \) at \( (12, 13) \).

Solution. (6) Evaluate the differential at \( x = 12, y = 13 \), \( dx = x - 12 \), \( dy = y - 13 \), \( dz = z - f(12, 13) \) at \( z - 5 \) and obtain

\[
z - 5 = \frac{12}{5} (x - 12) + \frac{13}{5} (y - 13),
\]

so that

\[
z = L(x, y) = 5 + \frac{12}{5} (x - 12) + \frac{13}{5} (y - 13)
\]

\[
= 5 - \frac{12}{5} x + \frac{13}{5} y + \frac{1}{5} (169 - 144)
\]

\[
= -\frac{12}{5} x + \frac{13}{5} y.
\]