Rules for Iterated Integrals

Rule 0. When an iterated integral is set up in a coordinate system, **ONLY** variables from that coordinate system are allowed.

(E.g., in polar coordinates, you can't have *x* or *y* present anywhere in the final iterated integral, only *r* and θ.)

Rule 1. A variable **NEVER** can be present in its own limits of integration, or in the limits of integration of a variable integrated after it is.

(So if we're integrating in the order $dx\,dz\,dy$, the limits on x can involve only *y* and *z*, the limits on z can involve only *y*, and the limits on y must be constants.)

Fact: Each limit of integration represents an equation, obtained from the variable being integrated equals its limits.

(In the integral $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{0}\int_{0}^{\sqrt{9-x^2-y^2}} \frac{z}{x^2+y^2+z^2}\,dz\,dy\,dx$, the limits represent the equations $z=0$, $z=\sqrt{9-x^2-y^2}$, $y=-\sqrt{9-x^2}$, $y=0$, $x=-3$ and $x=3$.)

Rule 2. **EVERY** limit of integration in an iterated integral **MUST** represent one of the following three types of equations:

Type 1. Actual boundaries of the region covered by this iterated integral.
Type 2. Equalities of the limits from an earlier integration in this iterated integral.
Type 3. Formal limits on a variable.

Formal limits are limits on the range of a variable due to the coordinate system being used. This can be because our differential requires it (e.g. in polar coordinates, $dA = r\,dr\,d\theta$ assumes $r \geq 0$), because going past it is unnecessary (and often anti-intuitive, e.g. in spherical coordinates we keep $\rho \geq 0$) or because going further will duplicate part of the region (we often keep $0 \leq \theta \leq 2\pi$ for this reason). In rectangular coordinates there are no formal limits except for $\pm\infty$, used in some improper integrals.

Important: Equations of Types 2 and 3 do not always get used, but those equations should not normally be between the limits that are used on a variable. Also, equations are <u>never</u> simultaneously Type 1 and Type 2.

Rule 3. Every actual piece (curve in 2D, surface in 3D) of the boundary of the region above **MUST** be present as a Type 1 limit in the iterated integral.

Critical consequence of rules 2 and 3: On the actual boundaries of the region, there can be at most two (there may be less, but only if there's also a formal limit) **ways to solve for the variable to be integrated first.** If you must integrate starting with a variable for which this is false, then you must do the multiple integral as a sum of two or more iterated integrals.