Summary on integration limit types in iterated integrals

Rectangular coordinates:

For proper integrals in rectangular coordinates, all limits of integration are either Type 1 or Type 2.

A limit is Type 1 if the equation of the variable to its limit is the equation of an actual side (curve bounding the area in 2D, surface bounding the volume in 3D) of the region that the integral is supposed to be over.

A limit is Type 2 if the equation of the variable to its limit provides a solution to the equality of the two limits to another integral in the iterated integral (the other integral must be inner to, i.e. evaluated before, the integral with the Type 2 limit). This means Type 2 limits are intersections of sides, but the point is that <u>not every intersection of sides is legitimate as a limit of integration</u> – **only** intersections that come up in this specific manner (equating the two limits of an inner integral) can be used for limits.

Polar, cylindrical and spherical coordinates:

In these coordinate systems, you can still get Type 1 and Type 2 limits, but you also can have Type 3 limits.

Type 3 limits are limits on the range of a variable due to the coordinate system being used. There are two subtypes, call them 3a and 3b. Type 3a could never appear as a Type 1 limit. The only possibilities that arise in Math 208 of Type 3a limits are the values $r = 0$, $\rho = 0$, $\phi = 0$, and $\phi = \pi$, the first in polar or cylindrical, the last 3 in spherical coordinates. (In improper integrals, a limit of ∞ can also be regarded as Type 3a.) Type 3b limits are <u>pairs</u> of limits that individually in some other iterated integrals would represent a Type 1 limit, but in the iterated integral at hand represent approaching an internal slice in the region from opposite sides. The only case we see in Math 208 is when θ runs through an interval of length 2π, most commonly the pair of limits $\theta = 0$ and $\theta = 2\pi$, but occasionally we can see something else like $\theta = -\pi$ and $\theta = \pi$ as a pair of limits. Note that both Type 3a and Type 3b limits are used only if needed – sometimes Type 1 or 2 limits stop a variables from reaching its possible Type 3 limit values (for example, a region where $r \geq 1$ everywhere can't use $r = 0$ as a Type 3 limit).

Type 1 and Type 2 limits arise as before, but some of the Type 2 limits are not intersections of sides. A Type 2 limit can now sometimes arise from the equality of a Type 1 or Type 2 limit with a Type 3a limit.

All coordinate systems - general rules

A limit is NEVER <u>both</u> Type 1 and Type 2, or Type 1 and Type 3a. On rare occasions, a limit can be partially Type 1 and partially Type 3b. Many iterated integrals in non-rectangular coordinates have some limits that are simultaneously Type 2 and Type 3 (either 3a or 3b).

NEVER go past a potential Type 2 limit unless you are sure that is the right thing to do. Usually, when you go past such a value, the limits reverse on the inner integral which gives rise to the potential Type 2 limit. If that reversal happens, and you need both parts, the quantity <u>must</u> be set up as a sum of iterated integrals.

ALWAYS make sure that each part of the actual sides of the region of integration comes from one of the Type 1 limits in the integral – otherwise that portion of the side is not actually bounding the region of the iterated integral. Conversely, when analyzing an iterated integral to understand the region of integration (usually before changing the order of integration or changing coordinate systems), be aware that any limits which are not Type 2 or 3 <u>must</u> be Type 1 – i.e., they give the real sides of the region.