AFTER CLASS NOTES FOR MATH 208
SECTION 006, FALL 2009

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FRIDAY, 10/30/09

Here is the integral that we worked on in class:

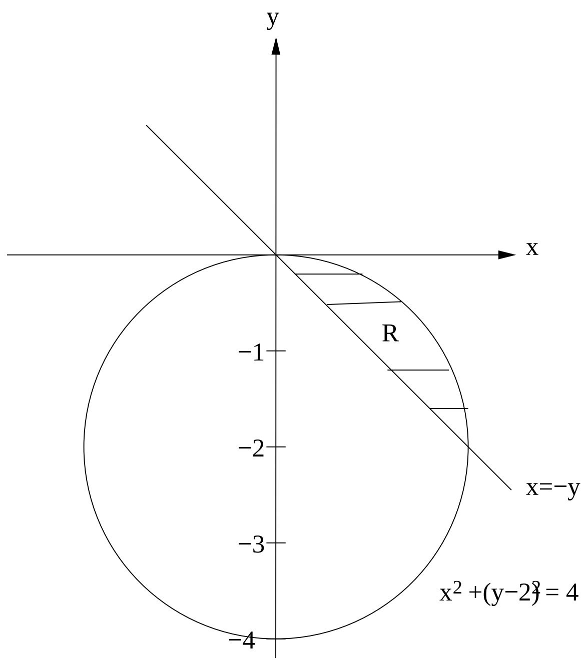
$$I = \int_{-2}^{0} \int_{2y^2}^{-4y} \int_{-y}^{\sqrt{z-y^2}} \frac{1}{x^2+y^2} dx\, dz\, dy.$$

First notice that the projection of the solid along the x-axis is a region in the yz-plane described by the outer two integrals. As y goes from -2 to 0, z runs from $z = 2y^2$ to $z = -4y$. Then, for a fixed y and z, thinking of the x-axis as "vertical", x is allowed to run from $x = -y$ to $x = \sqrt{z-y^2}$. That describes the solid D over which this becomes a triple integral. Furthermore, you see from the inner limits that two boundary surfaces are the bottom $x = -y$ and top $x = \sqrt{z-y^2}$, which up on squaring, becomes $x^2 + y^2 = z$, a paraboloid. These surfaces intersect at $-y = \sqrt{z-y^2}$, which upon squaring, gives $z = 2y^2$, the lower limit for z for a given y in the integral above. The cylinder $z = -4y$ is a "vertical side" relative to the x-axis. In summary, the surfaces that actually bound the solid are $x = -y$ and $z = x^2 + y^2$ and $z = -4y$.

Cylindrical coordinates look like a good candidate, so let's start by trying to get the integral expressed with z as the "vertical" variable, that is, on the inside. We have to figure out what the projection of D onto the xy-plane looks like. For starters, we know that y goes from -2 to 0. We're halfway there, sort of. For a fixed y, x runs from $x = -y$ to $x = \sqrt{z-y^2}$. OK, $x = -y$ is one boundary. What is the other? Well, plug in the limits for z and get one value $x = \sqrt{2y^2 - y^2} = \sqrt{y^2} = |y| = -y$, since $y \leq 0$ means that $|y| = -y$. We already have this limit. The other one is $x = \sqrt{-4y - y^2}$. So square both sides and obtain

$$\begin{aligned} x^2 &= -4y - y^2, \\ x^2 + y^2 + 4y &= 0 \\ x^2 + (y+2)^2 &= 4 = 2^2. \end{aligned}$$

This is a circle of radius 2 with center at $(0, -2)$. So x must start at $-y$ and finish on the circle we just described. Now we see that the projection R of D on the xy-plane looks like this:



Finally, for a given x, y, what are the limits on z as we run a vertical line through the solid at (x, y)? Look at the original integral form and you see that the upper limit for a give x, y of z is $z = -4y$, while the lower limit comes from the boundary surface $z = x^2 + y^2$. So here is the new integral as a double integral over R:

$$I = \iint_R \int_{x^2+y^2}^{-4y} \frac{1}{x^2+y^2} dz\, dA.$$

OK, now ask how we can sweep over the region in polar coordinates. On the outside, θ must go from $-\pi/4$ to 0. For a fixed θ, r must go from 0 to the boundary of the circle, which can be expressed in the form $r^2 = -4r\sin\theta$ or simply $r = -4\sin\theta$. Now use $2y^2 = 2r^2\sin^2\theta$ $-4y = -4r\sin\theta$ in polar coordinates and obtain

$$\begin{aligned}
I &= \int_{-\pi/4}^{0} \int_0^{-2\sin\theta} \int_{r^2}^{-4r\sin\theta} \frac{1}{r^2} dz\, r\, dr\, d\theta \\
&= \int_{-\pi/4}^{0} \int_0^{-4\sin\theta} (-4\sin\theta - r)\, dr\, d\theta \\
&= \int_{-\pi/4}^{0} \left(-4r\sin\theta - \frac{r^2}{2}\right) |_{r=0}^{-4\sin\theta} d\theta \\
&= \int_{-\pi/4}^{0} \left(16\sin^2\theta - 8\sin^2\theta\right) d\theta \\
&= 6\int_{-\pi/4}^{0} 8\left(\frac{1-\cos 2\theta}{2}\right) d\theta \\
&= \pi - 2.
\end{aligned}$$

Wednesday, 11/04/09

We ended the period with this example: Find the center of mass of a wire whose position vector is given by

$$\mathbf{r}(t) = \left\langle \sqrt{2}t, \sqrt{2}t, 4 - t^2 \right\rangle, \; 0 \le t \le 1,$$

and whose density function as a function of t is $\delta(t) = 3t$.

This amounts to the parametric representation

$$\begin{aligned} x(t) &= \sqrt{2}t \\ y(t) &= \sqrt{2}t \;\; 0 \le t \le 1, \\ z(t) &= 4 - t^2 \end{aligned}$$

so that

$$\begin{aligned} ds &= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt \\ &= \sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 + (-2t)^2}dt \\ &= 2\sqrt{1+t^2}dt. \end{aligned}$$

Hence

$$\begin{aligned} M &= \int_C \delta \, ds \\ &= \int_0^1 3t \cdot 2\sqrt{1+t^2}dt \\ &= \int_1^2 3u^{1/2}\, du \\ &= 3\frac{u^{3/2}}{3/2}\Big|_{u=1}^2 \\ & 2\left(2\sqrt{2} - 1\right), \end{aligned}$$

where the u-integral results from the substitution $u = 1 + t^2$.

Similarly, with the same substitution and observation that $t = \sqrt{u-1}$ on the interval of integration, we have

$$\begin{aligned} M_{yz} &= \int_C x\delta \, ds \\ &= \int_0^1 \sqrt{2}t \cdot 3t \cdot 2\sqrt{1+t^2}dt \\ &= 6\sqrt{2}\int_0^1 t^2\sqrt{1+t^2}\, dt \\ & 6\sqrt{2}\int_0^{\pi/4} \sec^3\theta \, d\theta \\ & \frac{3}{4}\sqrt{2}\left\{3\sqrt{2} + \ln\left(\sqrt{2} - 1\right)\right\} \end{aligned}$$

The third integral requires a trig substitution. Draw a right triangle with legs 1 and t. See page 461 of your text for a review. Finally, the fourth integral requires a trigonometric integration technique that you can see on page 459 of your text.

The integral for M_{xz} is the same as the integral for M_{yz}, so has the same answer. The integral

$$\begin{aligned} M_{xy} &= \int_C z\delta\, ds \\ &= \int_0^1 \left(4-t^2\right) 3t\sqrt{1+t^2}dt \\ &= \frac{3}{2}\int_0^1 (u-5)\, u^{1/2}\, du \\ &= \frac{1}{5}\left(38\sqrt{2}-22\right). \end{aligned}$$

is actually easier than the preceding integral, and can be worked out with the same substitution as in the mass integral.

Monday, November 16

We ended the period with an application of Green's theorem to area calculation: The problem is to compute the area of the astroid given parametrically by $\mathbf{r}(t) = \left\langle a\cos^3 t, a\sin^3 t\right\rangle$, $0 \leq t \leq 2\pi$. We used the vector field

$$F = \frac{1}{2}\langle -y, x\rangle = \langle M, N\rangle$$

and the flow (circulation) form of Green's theorem to obtain the area is

$$\iint_R 1\, dA = \iint_R \left(\frac{1}{2}+\frac{1}{2}\right) = \iint_R (N_x - M_y)\, dA] = \oint_C \frac{1}{2}(-y\, dx + x\, dy).$$

This integral is, by the parametrization $x = a\cos^3 t$, $dx = -3a\cos^2 t \sin t\, dt$, $y = a\sin^3 t$, $dy = 3a\sin^2 t \cos t\, dt$, so that

$$\begin{aligned}
\oint_C \frac{1}{2}(y\,dx - x\,dy) &= \frac{1}{2}\int_0^{2\pi} \left(-a\sin^3 t\,\left(-3a\cos^2 t\,\sin t\right) + a\cos^3 t\,3a\sin^2 t\,\cos t\right) dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} \left(\sin^4 t\,\cos^2 t + \cos^4 t\,\sin^2 t\right) dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} \sin^2 t\,\cos^2 t\left(\sin^2 t + \cos^2 t\right) dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} (\sin t\,\cos t)^2\, dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} \left(\frac{\sin 2t}{2}\right)^2 dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} \frac{1}{4}\left(\frac{1-\cos 4t}{2}\right) dt \\
&= \frac{3a^2}{8}\left(\frac{1}{2}t - \frac{\sin 4t}{8}\right)\Big|_0^{2\pi} \\
&= \frac{3}{8}\pi a^2.
\end{aligned}$$

1. What You Need to Know for Chapter 14 Integrations

Vector Operators: The "del" operator gives us a handy way to remember some basic operations by pretending that $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ (or its two-dimensional version $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$) is a vector, so for scalar function $f(x,y,z)$ and vector field $\mathbf{F}(x,y,z) = \langle M, N, P\rangle$ we have

$$\operatorname{grad}(\mathrm{f}) = \nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, N, P\rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle.$$

Differential Lengths: Given a curve C parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ $a \le t \le b$, the important differentials along the curve are dx, dy, dz, and ds (arc length), and important vectors are $d\mathbf{r}$, $\mathbf{T}(t)$ (unit tangent vector) and $\mathbf{n}(t)$ (unit normal vector), all of which are related by (last equation is for two-dimensional $\mathbf{r}(t) = \langle x(t), y(t)\rangle$)

$$dx = x'(t)\,dt, dy = y'(t)\,dt,\ dz = z'(t)\,dt$$

$$d\mathbf{r} = \langle dx, dy, dz\rangle = \mathbf{T}ds$$

$$ds = |d\mathbf{r}| = \sqrt{dx^2 + dy^2 + dz^2}$$

$$\mathbf{n}ds = \pm\langle dy, -dx\rangle.$$

An outward normal for a simple closed curve in the xy-plane traversed positively (counterclockwise) with respect to its interior would select the "+" sign in the formula for $\mathbf{n}$ds.

Use these differential formulas to turn line integrals into single integrals with the formulas

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy + P\,dz$$

$$\int_C g(x,y,z)\,dw = \int_a^b g(x(t), y(t), z(t))\,w'(t)\,dt.$$

Differential Areas: Given an orientable surface S with unit normal $\mathbf{n}$ and parametrized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v)\in R$, a region in the uv-plane, and given that $d\sigma$ is differential surface area on S and dA is differential area in the uv-plane, then

$$\mathbf{n}d\sigma = \pm\mathbf{r}_u \times \mathbf{r}_v dA$$

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

In the special case that S is the graph of $z = f(x,y)$, $(x,y)\in R$, a region in the xy-plane, then

$$\mathbf{n}d\sigma = \pm\langle -f_x, -f_y, 1\rangle\,dA$$

where a choice is made for the $\pm$ sign.

Use these differential formulas to turn a surface integral into a double integral with formulas

$$\iint_S g(x,y,z)\,d\sigma = \iint_R g(x(u,v), y(u,v), z(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_R \mathbf{F}(x(u,v), y(u,v), z(u,v))\cdot(\pm\mathbf{r}_u \times \mathbf{r}_v)\,dA.$$

Key Theorems about Differential Operators:

Each of these theorems is an analogue to the FTOC ($\int_a^b f'(x)\,dx = f(b) - f(a)$) for various differential operators.

Stokes' Theorem: Let S be an orientable piecewise smooth surface with normal $\mathbf{n}$ and simple closed curve C, positively oriented with respect to $\mathbf{n}$, as its boundary, and let $\mathbf{F}$ be a smooth vector field defined on S. Then

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma.$$

Note: "positively oriented" means that the direction of C is such that one can move in that direction while keeping the "forest of normals $\mathbf{n}$" on the left. If the surface is in the xy-plane interior to its boundary C and the normal is $\mathbf{k}$, then this simply amounts to orienting C counterclockwise.

Using $\mathbf{n} = \mathbf{k}$ in two dimensions, this specializes to:

Flow (tangential or circulation) form of Green's Theorem: Let C be a piecewise smooth simple closed curve enclosing the plane region R and positively oriented with respect to R with normal $\mathbf{k}$ and let $\mathbf{F} = \langle M, N \rangle$ be a smooth vector field defined on and inside C. Then the following two equivalent statements hold:

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA,$$

$$\int_C M\, dx + N\, dy = \iint_R (N_x - M_y)\, dA.$$

Gauss's Divergence Theorem: Let S be a closed orientable piecewise smooth surface enclosing the solid D with outward pointing normal $\mathbf{n}$ on S and let $\mathbf{F}$ be a smooth vector field defined on and inside S. Then

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

In two dimensions, this specializes to:

Flux (normal) form of Green's Theorem: Let C be a piecewise smooth simple closed curve enclosing the plane region R with outward pointing normal $\mathbf{n}$ on C and let $\mathbf{F} = \langle M, N \rangle$ be a smooth vector field defined on and inside C. Then the following two equivalent statements hold:

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \nabla \cdot \mathbf{F}\, dA,$$

$$\int_C M\, dy - N\, dx = \iint_R (M_x + N_y)\, dA.$$

Characterization of Conservative Vector Fields: Let D be a simply connected open set in 2D or 3D, $\mathbf{F} = \langle M, N, P \rangle$ a smooth vector field defined on D ($P = 0$ in the 2D case), and C an arbitrary curve contained in D. Then the following are equivalent:

(1) $\mathbf{F}$ is conservative, i.e., the line integral $\int \mathbf{F} \cdot d\mathbf{r}$ is path independent.

(2) $\mathbf{F}$ is a gradient field with (scalar) potential function f, that is, $\mathbf{F} = \nabla f$, in which case if the curve C starts at P and ends at Q, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P).$$

(3) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

(4) $\nabla \times \mathbf{F} = \mathbf{0}$ in D.