(24) 1. Let \( f(x, y, z) = 3x^2y - z \cos x \).

(a) Find \( \nabla f(0, 2, -1) \).

\[
\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle = \langle 6xy + z \sin(x), 3x^2 - \cos x \rangle
\]

So \( \nabla f(0, 2, -1) = \langle 0 + (-1) \cdot 0, 0, 0 \rangle = \langle 0, 0, 1 \rangle \)

(b) Find a unit vector in the direction of the maximum directional derivative of \( f \) at the point \((0, 2, -1)\).

\[
\text{The direction of maximum rate of change is That of } \nabla f. \text{ So The answer is } \frac{\nabla f(0, 2, -1)}{\| \nabla f(0, 2, -1) \|} = \langle 0, 0, 1 \rangle = \langle 0, 0, 1 \rangle
\]

(c) Find an equation for the tangent plane to the surface \( f(x, y, z) = 1 \) at the point \((0, 2, -1)\).

From part (a) The answer is

\[
0 \cdot (x-0) + 0 \cdot (y-2) - 1 \cdot (z+1) = 0
\]

That is, \( z = -1 \)

(18) 2. Given a function \( g(u, v) = f(x(u, v), y(u, v)) \) with all continuous partials, find \( \frac{\partial^2 g}{\partial u^2} \) in terms of \( f, x, y \) and their partials.

\[
\frac{\partial g}{\partial u} = \frac{\partial}{\partial u} f(x, y) = f_x \cdot x_u + f_y \cdot y_u
\]

\[
\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u} ( f_x \cdot x_u ) + \frac{\partial}{\partial u} ( f_y \cdot y_u ) = (f_{xx} \cdot x_{uu} + f_{xy} \cdot y_{uu})x_u + f_x \cdot x_{uu} + (f_{yx} \cdot x_{uu} + f_{yy} \cdot y_{uu})y_u
\]

\[
= f_{xx} \cdot x_u^2 + 2f_{xy} \cdot x_u y_u + f_x \cdot x_{uu} + f_{yx} \cdot x_{uu} + f_{yy} \cdot y_u^2 + f_y \cdot y_{uu}
\]
Let \( f(x,y) = 3x^2 + y^2 - 9x + 4y \).

(a) Find all critical points of \( f \). If \( f \) is smooth, so

\[
\begin{align*}
0 & = f_x = 6x - 9 \\
0 & = f_y = 2y + 4
\end{align*}
\]

Get \( y = -2 \), and \( x^2 = 1 \), so \( x = \pm 1 \).

Critical points: \((1, -2), (-1, -2)\).

(b) Use the second derivative test to classify the critical points of \( f \).

\[
\begin{align*}
f_{xx} & = 18, \\
f_{yy} & = 2, \\
f_{xy} & = 0.
\end{align*}
\]

So \( Df(a, b) = f_{xx} f_{yy} - f_{xy}^2 = 36 \times 2 > 0 \). Now appropriate maximum/minimum here.

But \( f_{yy} (1, 2) = 2 > 0 \), so it is a local min. \( Df(1, 2) = -36 < 0 \). So have a saddle pt. here.

(c) Suppose the domain of \( f(x, y) \) is restricted to the square \( 0 \leq x, y \leq 2 \). Does \( f(x, y) \) have a global maximum or minimum on this domain? In either case, explain your answer and briefly indicate how you would find them if an answer is affirmative.

The set is closed and bounded, hence must assume its global max/min in the set, namely at interior critical points or on boundary.

So we would look at critical points interior to the square, then search along the four line boundaries for extrema using 1 variable methods finally, we would evaluate \( f \) at all these candidate points and pick extrema.
(25) Let \( f(x, y) = 4xy \).
(a) Find the extrema of \( f \) subject to the constraint \( 4x^2 + y^2 = 8 \) by the method of Lagrange multipliers.

Let \( q(x, y) = 4x^2 + y^2 - 8 \) so constraint is \( q(x, y) = 0 \).

L.M. Equations: \( \langle f_x, f_y \rangle = \lambda \langle q_x, q_y \rangle \), \( \lambda = 0 \),

i.e.) \[
\begin{align*}
4y &= 8\lambda x, \\
4x &= 2y \\
4x^2 + y^2 &= 8.
\end{align*}
\]

\( x, y \neq 0 \) so \( \frac{4y}{8x} = \lambda = \frac{4x}{2y} \), so \( \frac{4}{8} = \frac{2x}{y} \), i.e. \( y^2 = 4x^2 \).

Thus \( 4x^2 + y^2 = 2y^2 = 8 \), \( y^2 = 4 \), \( y = \pm 2 \).

For each \( y \), \( x^2 = \frac{y^2}{4} = 1 \), so \( x = \pm 1 \).

Get 4 critical points. Evaluate to find extreme values:

\[
\begin{align*}
f(1, 2) &= 4 \cdot 2 = 8 \\
f(-1, 2) &= -4 \cdot 2 = -8 \\
f(1, -2) &= -4 \cdot 2 = -8 \\
f(-1, -2) &= 4 \cdot 2 = 8
\end{align*}
\]

So \( 8 \) is maximum value and \(-8\) is minimum value of \( f \), subject to \( g = 0 \).

(b) Find the absolute extrema of \( f \) over the region \( 4x^2 + y^2 \leq 8 \).

Use EVT:

In addition to above, have to find all critical interior pts of \( f(x, y) \) and test them.

\[
\begin{align*}
f_x &= 4y = 0 \text{ gives } x = y = 0. \\
f_y &= 4x = 0
\end{align*}
\]

But \( f(0, 0) = 0 \) which is neither max nor min.

So max & min values over whole region \( 4x^2 + y^2 \leq 8 \)
are same as in (a): Max value of \( f \): 8
Min value of \( f \): -8