# The differentials for flux integrals
## Math 208

Our text fails to explicitly state the formula for $\mathbf{n}\,d\sigma$ when the surface is written as one variable is a function of the other two. This is by far the most common case used, and knowing $\mathbf{n}\,d\sigma$ saves several steps, so it should be known. Most often, $z$ is a function of $x$ and $y$. In that case we get:

If the surface is part of $z = f(x,y)$, then $\mathbf{n}\,d\sigma = \pm(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dy\,dx$.

The choice of sign depends on the orientation of the surface involved, + giving downward orientation, − giving upward (since up/down is determined by the **k** coefficient). Also, be aware that the integral can, of course, be done $dy\,dx$ as indicated, $dx\,dy$, or even $r\,dr\,d\theta$. But treating $\mathbf{n}\,d\sigma$ as a combination and taking the dot product with the vector field $\mathbf{F}$ is almost always simpler than finding $\mathbf{n}$ and $d\sigma$ separately, calculating $\mathbf{F}\bullet\mathbf{n}$, and then multiplying by $d\sigma$. For example:

**Example:** Find the flux of $\mathbf{F}(x,y,z) = \langle 3x, 3y, -2z \rangle$ over that portion of the upward oriented paraboloid $x^2 + y^2 - z = 0$ which satisfies $z \le 9$.

**Solution:** The surface equation, solved for $z$, gives us $z = x^2 + y^2$ so we need $f(x,y) = x^2 + y^2$. Since we want upward oriented,

$\mathbf{n}\,d\sigma = -(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k})\,dy\,dx = \langle -2x, -2y, 1 \rangle\,dy\,dx$, which leads to

$$\begin{aligned}
\iint_S \mathbf{F}\bullet\mathbf{n}\,d\sigma &= \iint_{R_{xy}} \langle 3x, 3y, -2z \rangle \bullet \langle -2x, -2y, 1 \rangle\,dy\,dx \\
&= \iint_{R_{xy}} (-6x^2 - 6y^2 - 2z)\,dy\,dx \\
&= \iint_{R_{xy}} (-8x^2 - 8y^2)\,dy\,dx
\end{aligned}$$

But $R_{xy}$ is the region in the $xy$-plane where $z = x^2 + y^2 \le 9$, which means this integral is best done in polar coordinates. We get:

$$\begin{aligned}
\iint_{R_{xy}} (-8x^2 - 8y^2)\,dy\,dx &= \int_0^{2\pi}\int_0^3 -8r^3\,dr\,d\theta \\
&= \left(-2r^4\Big|_0^3\right)\left(\theta\Big|_0^{2\pi}\right) = -324\pi
\end{aligned}$$

Note that one can "rotate" the roles of the variables, and get corresponding formulas:

If the surface is part of $x = f(y,z)$, then $\mathbf{n}\,d\sigma = \pm(-\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k})\,dy\,dz$.

and likewise

If the surface is part of $y = f(x,z)$, then $\mathbf{n}\,d\sigma = \pm(f_x\mathbf{i} - \mathbf{j} + f_z\mathbf{k})\,dx\,dz$.

Also note that the formulas for $d\sigma$ in these settings (which are in the text at the bottom of p. 895 and top of p. 896) are just the formulas for the lengths of these vector differentials. In general, if you find yourself having trouble memorizing all of the differentials for surface integrals, memorize the ones for $\mathbf{n}\,d\sigma$, and if you're doing a surface integral which is not a flux integral, find $d\sigma$ by taking the length of the vector part. The formula for $\mathbf{n}\,d\sigma$ is generally simpler to memorize and use than the formulas for $\mathbf{n}$ and $d\sigma$ done separately. (When done separately, some extra square roots arise, which eventually cancel.) For $\mathbf{n}\,d\sigma$ we mainly use the above cases and the parametric case, where again the combined differential is simpler to use than $\mathbf{n}$ and $d\sigma$ separately, but it is not given explicitly in the text:

If the surface is given parametrically by $\mathbf{r}(u,v)$, then $\mathbf{n}\,d\sigma = \pm(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$.

The formulas given in the book for the level surface $g(x,y,z) = c$ are generally harder to use than the formulas above, and the surfaces we give you are always either easy to parametrize or easy to solve for one of the variables in terms of the other two, so if you know the above formulas and know how to parametrize standard surfaces, you're generally covered.

Exercises:

1. Find the flux of $\mathbf{F}(x,y,z) = <2z, 0, 4y^2>$ over the upward oriented portion of $z = 2x + y^2 + 3$ that is defined by $-1 \le x \le 3$ and $1 \le y \le 4$.
2. Find the flux of $\mathbf{F}(x,y,z) = <3,1,-2>$ over the portion of the surface $y + z^2 = x^2$ that has $-2 \le x \le 2$, $-2 \le z \le 4$ and is oriented toward decreasing *y*.
3. Find the flux of $\mathbf{F}(x,y,z) = <2x, y, z>$ over the finite piece of $x = y^2 z^4$ bounded by $z = -2$, $y = -z$, and $y = 2z$, if the orientation is away from you as viewed from a point on the negative *x*-axis.

Answers:

1. $-240$
2. 72
3. 384