Direct Calculation Methods for Integrals

1. **Definite Integral**: $\int_a^b f(x) \, dx$. Here $a$ and $b$ could involve other variables besides $x$. Use Calculus I and II to solve these.

2. **Line Integral**: $\int_C M \, dx + N \, dy + P \, dz$ or $\int_C f \, ds$. Use this three-step procedure:
   - (a) Parametrize $C$ in terms of $t$, say, with $a \leq t \leq b$.
   - (b) Compute differentials $dx$, $dy$, $dz$, $ds$ as needed.
   - (c) Reduce line integral to a definite integral on interval $a \leq t \leq b$ by substitution.

3. **Iterated Integral**: $\int_0^b \int_{f(v)}^{g(v)} F(u,v) \, du \, dv$ or $\int_a^b \int_{g(w)}^{h(w)} F(u,v,w) \, du \, dv \, dw$. These are just definite integrals two or three times.

4. **Double or Triple Integral**: $\iiint_D f \, dV$ or $\iint_D f \, dV$. Use a Fubini-type theorem to reduce these to iterated integrals.

5. **Surface Integrals**: $\int_S f \, d\sigma$. Use this three-step procedure:
   - (a) Parametrize $S$ in terms of $u,v$, say, with $(u,v) \in R$, a region in the $uv$-plane and write out a position vector $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ for points on $S$.
   - (b) Compute differential $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| \, dA$, where $dA$ is differential area in $uv$-plane.
     Important special case where this is “precomputed.” If $S$ is the graph of $z = f(x,y)$, then $d\sigma = \sqrt{f_x^2 + f_y^2 + 1} \, dA$, where $dA$ is differential area in the $xy$-plane.
   - (c) Reduce surface integral to a double integral over $R$ by substitution.

6. **Flux Integral**: $\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$. Here $\mathbf{n}$ is a continuous unit normal vector to $S$. Use same three-step procedure as in 5, except that in (b) $\mathbf{n} \, d\sigma = \pm \mathbf{r}_u \times \mathbf{r}_v \, dA$.
   Important special case where this is “precomputed.” If $S$ is the graph of $z = f(x,y)$, then $\mathbf{n} \, d\sigma = \pm (-f_x, -f_y, 1) \, dA$, where $dA$ is differential area in the $xy$-plane.

Indirect Calculation Methods for Integrals

The idea is to indirectly calculate one side by directly calculating the other side of one of the following theorems.

1. (Flux integrals) **Gauss Divergence Theorem** in 3-D:

   $$ \iiint_D \nabla \cdot \mathbf{F} \, dV $$

   Here the boundary of the solid $D$ is the (closed) surface $S = \partial D$ with outward pointing normal $\mathbf{n}$.

2. Specialize the Divergence Theorem to 2-D in $xy$-plane and obtain the flux form of Green’s Theorem for boundary closed curve $C = \partial R$ positively oriented with respect to the region $R$ in the $xy$-plane:

   $$ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R M \, dy - N \, dx = \iint_R (M_x + N_y) \, dA = \iint_R \nabla \cdot \mathbf{F} \, dA $$

3. (Flow Integrals) **Stokes’ Theorem**:

   $$ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma $$

   Here $C = \partial S$ is the closed boundary of the surface $S$, positively oriented with respect to the surface normal $\mathbf{n}$.

4. Specialize Stokes’ Theorem to 2-D in $xy$-plane and obtain the flow form of Green’s Theorem for boundary closed curve $C = \partial R$ positively oriented with respect to the region $R$ in the $xy$-plane:

   $$ \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R M \, dx + N \, dy = \iint_R (N_x - M_y) \, dA = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA $$