Direct Calculation Methods for Integrals

1. *Definite Integral:* $\int_a^b f(x)\,dx$. Here a and b could involve other variables besides x. Use Calculus I and II to solve these.

2. *Line Integral:* $\int_C M\,dx + N\,dy + P\,dz$ or $\int_C f\,ds$. Use this three-step procedure:

   (a) Parametrize C in terms of t, say, with $a \le t \le b$.

   (b) Compute differentials dx, dy, dz, ds as needed.

   (c) Reduce line integral to a definite integral on interval $a \le t \le b$ by substitution.

3. *Iterated Integral:* $\int_a^b \int_{f(v)}^{g(v)} F(u,v)\,du\,dv$ or $\int_a^b \int_{f(w)}^{g(w)} \int_{g(v,w)}^{h(v,w)} F(u,v,w)\,du\,dv\,dw$. These are just definite integrals two or three times.

4. *Double or Triple Integral:* $\iint_R f\,dA$ or $\iiint_D f\,dV$. Use a Fubini-type theorem to reduce these to iterated integrals.

5. *Surface Integrals:* $\iint_S f\,d\sigma$. Use this three-step procedure:

   (a) Parametrize S in terms of u, v, say, with $(u,v) \in R$, a region in the uv-plane and write out a position vector $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for points on S.

   (b) Compute differential $d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,dA$, where dA is differential area in uv-plane.
   Important special case where this is "precomputed:" If S is the graph of $z = f(x,y)$, then $d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\,dA$, where dA is differential area in the xy-plane.

   (c) Reduce surface integral to a double integral over R by substitution.

6. *Flux Integral:* $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$. Here $\mathbf{n}$ is a continuous unit normal vector to S. Use same three-step procedure as in 5, except that in (b) $\mathbf{n}\,d\sigma = \pm\mathbf{r}_u \times \mathbf{r}_v\,dA$.
   Important special case where this is "precomputed:" If S is the graph of $z = f(x,y)$, then $\mathbf{n}\,d\sigma = \pm\langle -f_x, -f_y, 1\rangle\,dA$, where dA is differential area in the xy-plane.

Indirect Calculation Methods for Integrals

The idea is to indirectly calculate one side by directly calculating the other side of one of the following theorems.

1. (Flux integrals) **Gauss Divergence Theorem** in 3-D:
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV$$
   Here the boundary of the solid D is the (closed) surface $S = \partial D$ with outward pointing normal $\mathbf{n}$.

2. Specialize the Divergence Theorem to 2-D in xy-plane and obtain the **flux form of Green's Theorem** for boundary closed curve $C = \partial R$ positively oriented with respect to the region R in the xy-plane:
$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R (M_x + N_y)\,dA = \iint_R \nabla\cdot\mathbf{F}\,dA$$

3. (Flow Integrals) **Stokes' Theorem**:
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma$$
   Here $C = \partial S$ is the closed boundary of the surface S, positively oriented with respect to the surface normal $\mathbf{n}$.

4. Specialize Stokes' Theorem to 2-D in xy-plane and obtain the **flow form of Green's Theorem** for boundary closed curve $C = \partial R$ positively oriented with respect to the region R in the xy-plane:
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA = \iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}\,dA$$