Final Exam **Math 314, Section 006** **Fall 2006**

Name:______________________________ **Score:**____________

Instructions: Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate.* You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g., π) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

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(30) **1.** In the following problem use the fact that R is the reduced row-echelon form of the augmented matrix $\widetilde{A} = [A \,|\, \mathbf{b}]$ where

$$\widetilde{A} = \begin{bmatrix} 3 & 1 & -2 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 2 \\ 3 & 2 & -1 & 1 & 1 & 9 \\ 0 & 2 & 2 & -1 & 1 & 8 \end{bmatrix} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v_3}, \mathbf{v}_4, \mathbf{v}_5, \mathbf{b}] \text{ and } R = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

(a) Find a basis for the row space of A and the dimension of the row space.

(b) Find a basis for the column space of A and the rank of A.

(c) Find a basis for the null space of A and the nullity of A.

(d) Find the general solution to the system $A\mathbf{x} = \mathbf{b}$.

(e) Fill in the blanks below if possible or give a reason if not possible. (Hint: $A\mathbf{x}$ as a l.c.)

$$\_\_\_\_\_\_\mathbf{v}_3 \;+\; \_\_\_\_\_\_\mathbf{v}_4 \;+\; \_\_\_\_\_\_\mathbf{v}_5 \;=\; \mathbf{b}$$

(18) **2.** Calculate A^{-1} and $\det(A)$, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 2 & 4 & 2 \end{bmatrix}$ and use A^{-1} to solve the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (2, 0, 4)$.

(16) **3.** Let $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$, so that $R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is the reduced row echelon form of $[A, B]$ (assume this). Find bases and dimensions of the following subspaces of $\mathbb{R}^3$:

(a) $\mathcal{C}(A)$

(b) $\mathcal{C}(B)$

(c) $\mathcal{C}(A) + \mathcal{C}(B)$

(d) $\mathcal{C}(A) \cap \mathcal{C}(B)$

(16) **4.** Let $\mathbf{w}_1 = (1, 0, 1, 0)$, $\mathbf{w}_2 = (0, 1, 0, 1)$ and $\mathbf{w}_3 = (0, 1, 0, -1)$.
(a) Show these vectors are orthogonal in the inner product space $\mathbb{R}^4$ with the standard inner product.

(b) Why are these vectors linearly independent?

(c) Are these vectors are orthogonal in the inner product space $\mathbb{R}^4$ with the non-standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 + x_4 y_4$? If not, use Gram-Schmidt to generate an orthogonal set from them.

(16) **5.** Are following subsets W of vector space V are subspaces of V? Justify your answers.

(a) $W = \left\{ \mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x}^T A \mathbf{x} = -1 \right\} \subseteq V = \mathbb{R}^3$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) $W = \{ \mathbf{x} \in \mathbb{R}^2 \,|\, A\mathbf{x} = 0 \} \subseteq V = \mathbb{R}^2$, where $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

(16) **6.** The matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is symmetric with an eigenvector $\mathbf{v}_1 = (1, 1)$.

(a) Find an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A and find the eigenvalues of A.

(b) Use (a) to find a matrix formula (with no matrix products) for A^k, $k > 0$.

(20) **7.** For the matrix $A = \begin{bmatrix} 1 & 8 & 8 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (You do *not* have to find P^{-1}.)

(16) **8.** Find a least squares solution to the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Is the least squares solution a genuine solution?

(22) **9.** Fill in the blanks or answer True/False (T/F).

(a) If A, B are 2×2 matrices, then $(AB)^2 = A^2B^2$ (T/F)____________.

(b) Every orthonormal set of vectors is linearly independent (T/F) ____________.

(c) If A is real symmetric, then the eigenvalues of A are real (T/F) ____________.

(d) If the linear system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions and A is an $m \times n$ matrix, then $n > m$ (T/F) ____________.

(e) If $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\dim V = 3$ (T/F) ____________.

(f) The CBS inequality for an inner product space V says that for all vectors $\mathbf{u}, \mathbf{v} \in V$, ____________.

(g) If x solves the normal equations for $A\mathbf{x} = \mathbf{b}$ then $A\mathbf{x}$ is the projection of $\mathbf{b}$ into ____________.

(h) The cosine of the angle between $(1, 1)$ and $(1, -3)$ in $\mathbb{R}^2$ with the standard inner product is ____________.

(i) $\begin{bmatrix} 1 & 2 \\ 0 & 1+i \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & i & 1 \end{bmatrix} =$

(j) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}^T =$

(k) $T((x, y)) = (x + y, 2x, 4y - x)$ is a matrix multiplication operator $T_A((x, y))$, where $A =$____________.

(30) **10.** Give brief answers to the following. Do only one (honors students, two) of (d), (e), or (f).
(a) Let $f(x) = x$ and $g(x) = x^2$ in the inner product space $C[0,1]$ with the standard function space inner product ($\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$. Find the projection of $g(x)$ along $f(x)$ in this space.

(b) Use determinants to determine for what x the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 4 & x \end{bmatrix}$ is singular.

(c) Find $\|2(\mathbf{u} - \mathbf{v})\|_p$, $p = 1, 2, \infty$, where $\mathbf{u} = (-2, 1, 1)$ and $\mathbf{v} = (0, 3, -3)$

(d) Show *from definition* that if λ is an eigenvalue of invertible A, then $1/\lambda$ is an eigenvalue of A^{-1}.

(e) Show *from definition* that if $\mathbf{v}_1 = \mathbf{0}$ then the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in the vector space V is linearly dependent.

(f) Show that if $\mathbf{v} \in \mathbb{R}^n$ is a nonzero vector, then the matrix $H = I_n - 2\dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$ is orthogonal.