Linear Algebra Report **Due: Some Date**

Subject: Management of Domestic Sheep Populations

Description of the problem: You are working for the New Zealand Department of Agriculture on a project for sheep farmers. The species of sheep that these farmers raise have a lifespan of 12 years. Of course, some live longer but they are sufficiently few in number that they may be ignored in your population study. Accordingly, you divide sheep into 12 age classes, namely those in the first year of life, etc. You have conducted an extensive survey of the demographics of this species of sheep and obtained the following information about the demographic parameters a_i and b_i, where a_i is the reproductive rate for sheep in the ith age class and b_i is the survival rate for sheep in that age class (i.e., the fraction of sheep in that age class that survive to the $i + 1$th class.)

i	1	2	3	4	5	6	7	8	9	10	11	12
a_i	.000	.023	.145	.236	.242	.273	.271	.251	.234	.229	.216	.210
b_i	.845	.975	.965	.950	.926	.895	.850	.786	.691	.561	.370	-

The problem is as follows: in order to maintain a constant population of sheep, farmers will harvest a certain number of sheep each year. Harvesting need not mean slaughter; it can be accomplished by selling animals to other farmers, for example. It simply means removing sheep from the population. Denote the fraction of sheep which are removed from the ith age group at the end of each growth period (a year in our case) by h_i. If these numbers are constant from year to year, they contsitute a *harvesting policy.* If, moreover, the yield of each harvest, i.e., total number of animals harvested each year, is a constant and the age distribution of the remaining populace is essentially constant after each harvest, then the harvesting policy is called *sustainable.* If all the h_i's are the same, say h, then the harvesting policy is called *uniform.* An advantage of uniform policies is that they are simple to implement: one selects the sheep to be harvested at random.

Your problem is as follows: find a uniform sustainable harvesting policy to recommend to farmers, and find the resulting distribution of sheep that they can expect with this policy. Farmers who raise sheep for sale to markets are also interested in a sustainable policy that gives a maximum yield. If you can find such a policy that has a larger annual yield than the uniform policy, then recommend it. On the other hand, farmers who raise sheep for their wool may prefer to minimize the annual yield. If you can find a sustainable policy whose yield is smaller than that of the uniform policy, make a recommendation accordingly. In each case find the expected distribution of your harvesting policies. Organize your results for a report to be read by your supervisor and an informed public.

Procedure: Express this problem as a discrete linear dynamical system $x^{k+1} = Lx^k$, where L is a Leslie matrix of the form

$$L = \begin{bmatrix} a_1 & a_2 & a_3 & \cdot & a_{n-1} & a_n \\ b_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1} & 0 \end{bmatrix}.$$

The facts you need to know (and may assume as standard facts about Leslie matrices) are as follows: Such a matrix will have exactly one positive eigenvalue which turns out to be a simple eigenvalue (not repeated). Moreover, if at least two adjacent entries of the first row are positive, this eigenvalue will be a *dominant* eigenvalue, i.e., it is strictly larger than any other eigenvalue in absolute value. In particular, if the positive eigenvalue is 1, then starting from any nonzero initial state with non-negative entries, successive states converge to an eigenvector belonging the the eigenvalue 1 which has all non-negative entries. Scale this vector by dividing it by the sum of its components and one obtains an eigenvector which is a probability distribution vector, i.e., its entries are non-negative and sum to 1. The entries of this vector give the long term distribution of the population in the various age classes.

In regards to harvesting, let H be a diagonal matrix with the harvest fractions h_i down the diagonal. (Here $0 \leq h_i \leq 1$.) Then the population that results from this harvesting at the end of each period is given by $x^{k+1} = Lx^k - HLx^k = (I - H)Lx^k$. But the matrix $(I - H)L$ is itself a Leslie matrix, so the theory applies it as well. There are other theoretical tools, but all you need to do is to find a matrix $H = hI$ so that 1 is the positive eigenvalue of $(I - H)L$. You can do this by trial and error, a method which is applicable to any harvesting policy, uniform or not. However, in the case of uniform policies it's simpler to note that $(I - H)L = (1 - h)L$, where h is the diagonal entry of H.

Implementation Notes: You will need a technology tool for this report. Be sure to include implementation details of this tool in your report.

About reports: Refer to page 61 of our text for advice on how to write up a project.