Linear Algebra Project Due: Some Date
Subject: Heat Flow in an Insulated Rod Points: 20

Description of the problem: You are working for the firm Universal Dynamics on a project which has a number of components. You have been assigned the analysis of a component which is similar to a laterally insulated rod. The problem you are concerned with is as follows: part of the specs for the rod are that no material which will stay at temperatures above 60 degrees Celcius for a long period of time at any point will be acceptable. You must decide if any of the materials listed below are acceptable and write a report on your findings. You may assume that the rod is of unit length. Suppose further that internal heat sources come from a position dependent function $f(x)$, $0 \leq x \leq 1$ and that heat is also generated at each point in amounts proportional to the temperature at the point. Also suppose that the left and right ends of the rod are held at 0 and 50 degrees Celcius, respectively. When sufficient time passes, the temperature of the rod at each point will settle down to "steady state" values, dependent only on position x. These are the temperatures you are interested in. It can be shown from physical principles that if u is the temperature function of the rod, i.e., $u(x)$ is the temperature of the rod at x, then $u(x)$ and $f(x)$ are related by the following boundary value problem (BVP):

$$\begin{aligned} -k\, d^2u/dx^2 + s\, u(x) &= f(x), \\ u(0) &= 0, \\ u(1) &= 50. \end{aligned}$$

Here k is a constant, called the *thermal conductivity* of the rod, associated with the material of the rod and s is another constant. For your problem take $f(x) = 200\cos(x^2)$ and $s = 0.4$. Here are the conductivity constants for the materials which your company is considering building the rod with. Which are acceptable?

Platinum: $k = .17$
Zinc: $k = .30$
Aluminum: $k = .50$
Gold: $k = .75$
Silver: $k = 1.00$

Procedure: For the solution of the problem, formulate a discrete approximation to the BVP similarly to what was done in class. Choose an integer n and divide the interval [0, 1] into $n + 1$ equal subintervals with endpoints $0 = x_0, x_1, ..., x_{n+1} = 1$. Then the width of each subinterval is $h = 1/(n + 1)$. Next let u_i be our approximation to $u(x_i)$ and replace the second derivative in the equation $-u''(x_i) + (s/k)u(x_i) = f(x_i)/k$, $i = 1, ...n$ by the the approximation to the derivative $u''(x_i) \approx (u_{i-1} - 2u_i + u_{i+1})/h^2$. There results a linear system of n equations in the n unknowns $u_1, u_2, ..., u_n$.

For this problem divide the rod into 15 equally sized subintervals and take $n = 14$. Use the largest u_i as an estimate of the highest temperature at any point in the rod. You might want to use a larger n, too, to corroborate your findings.

Implementation Notes: Set up the coefficient matrix a and right hand side b for the system. The coefficient matrix should be a tridiagonal matrix with $(2 + h^2 s/k)$'s down the main diagonal, -1's in the two off diagonals and 0's elsewhere.

About reports: The first thing you need to know about report writing is the intended audience. You may assume that this report will be read by your bosses, who are technical people such as yourself. Therefore, you should write a brief statement of the problem and discussion of methodology. You may assume the physics of this problem is as given above without further justification, but in real life you would be expected to offer some explanation of physical principles you employ. Another good thing to have in mind is a target length for your paper. Do not clutter your work with long lists of numbers and try to keep the length of this report at 4 or fewer pages of text.

Most kinds of discourse should have three parts: a beginning, a middle and an end. Roughly, a beginning should consist of introductory material. In the middle you develop the ideas described or theses proposed in the introduction, and in the end you summarize and tie up loose ends. Of course, rules about paper writing are not set in concrete, and authors vary on exactly how they organize papers of a given kind. Also, a part could be quite short; for example, an introduction might only be a paragraph or two. Here is a sample skeleton for a report (perhaps rather more elaborate than you need): 1. Introduction (title page, summary and conclusions), 2. Main Sections (problem statement, assumptions, methodology, results, conclusions), 3. Appendices (such as mathematical analysis, graphs, possible extensions, etc) and References.

References: 1. Thomas Shores, *Applied Linear Algebra and Matrix Analysis*, 2nd ed, New York: Springer-Verlag, Publishers, 2018.

2. Eric Becker, Graham Carey and J. Tinsley, *Finite Elements: An Introduction, Volume I*, Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1981, pp. 42-47.

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