JDEP 384H: Numerical Methods in Business

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Department of Mathematics

Lecture 11, February 13, 2007
110 Kaufmann Center

Outline

1. Basic Financial Assets and Related Issues

2. BT 2.4: Derivatives
   - The Basics
   - Black-Scholes

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The Problem with American Options

The Heart of the Difficulty:

- We enforce an inequality, e.g., $P(S,t) \geq \max\{K-S, 0\}$ that fights with the Black-Scholes equation.
- The new problem becomes what is called a linear complementarity problem: Solve

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0$$

  subject to the constraint $P(S,t) \geq \max\{K-S, 0\}$ where if one inequality is strict, then the other is an equality.
- There are no closed form solutions to these problems. They need advanced numerical methods of approximation.

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Enter Binomial Lattices

There is a workaround that is sometimes effective:

Discretize the process of option pricing into small time intervals and treat the pricing problem like a binomial lattice, i.e.,

- Track the price $S(t)$ of a stock, which we think of as a random variable, through time with some simplifying assumptions:
- Times measured are discrete in a fixed unit (period),e.g., $\delta t = 1/365$, i.e., days. Thus, the only values of $S(t)$ are $S = S_0 = S(0) > 0$, $S_1 = S(\delta t)$, ..., $S_n = S(n \cdot \delta t)$.
- At each stage, the stock will either move up with a return of u or move down with a return of d and does so with probabilities p and $q = 1 - p$.
- Stock price goes up or down by factors, i.e., a value S_n becomes $S_{n+1} = S_n \cdot u$ or $S_{n+1} = S_n \cdot d$ at the next time step.

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About "Up", "Down", and "Probabilities":

- The probabilities p and $q = 1 - p$ are "artificial" probabilities that are "risk-neutral" in the sense that

$$E\left[S_{n+1}\right] = pS_n u + (1-p)\, S_n d = S_n e^{r \cdot \delta t}$$

  when expectation is taken with respect to them.
- We have three parameters to determine: u, d, and p.
- One can deduce: $p = \left(e^{r \cdot \delta t} - d\right) / (u - d)$, the so-called **risk-neutral probability** by a hedging argument, as with Black-Scholes, provided that one can replicate option payoffs with "primary" assets (stocks and bonds), that is, the market is **complete**.
- An important fact from economics: if there is no arbitrage, risk-neutral probabilities exist, and if the market is complete, these probabilities are unique.
- What about u and d?

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Details of a Binomial Lattice for Options: II

More Assumptions:

- The variance of the new stock price should be $S_n^2\sigma^2 \cdot \delta t$. Why? (Start with $S_{n+1} \approx S_n + dS_n$.)
- This leads to a nonlinear equation in u and d. We need one more condition to determine the three parameters.
- We take the CRR (Cox, Ross and Rubinstein) approach: $d = 1/u$ and, of course, $u > 1$.
- This leads to a nonlinear equation in u:

$$e^{r\cdot\delta t}\left(u + \frac{1}{u}\right) - 1 - e^{2r\cdot\delta t} = \sigma^2 \cdot \delta t$$

  with approximate solution $u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}}$.
- There are others such as the Jarrow-Rudd parameterization $p = \frac{1}{2}$, which leads to formulas $p = \dfrac{1}{2}$, $u = e^{\left(r-\frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t}}$, $d == e^{\left(r-\frac{\sigma^2}{2}\right)\delta t - \sigma\sqrt{\delta t}}$.

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Sidebar: Dividend Paying Assets

If dividends are involved:

- We still assume $d = 1/u$ and, of course, $u > 1$.
- As before, one can deduce the risk-free probability: $p = \left(e^{r \cdot \delta t} - d\right) / (u - d)$ by the same hedging argument.
- So value of an option f_n satisfies $e^{r \cdot \delta t} f_n = p \cdot f_u + (1 - p) f_d$
- Expected value of S_{n+1} is risk-free, but must account for the payment of dividends:

$$E\left[S_{n+1}\right] = pS_n u + (1 - p) S_n d = S_n e^{(r - D_0) \cdot \delta t}.$$

- The variance of the stock price should be $S_n^2 \sigma^2 \cdot \delta t$.
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Solution: $u = \frac{M + \sqrt{M^2 - 4}}{2}$, $M = 1 + \sigma^2 \cdot \delta t + e^{2(r - D_0) \cdot \delta t}$.

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Solution: $u = \frac{M+\sqrt{M^2-4}}{2}$, $M = 1 + \sigma^2 \cdot \delta t + e^{2(r-D_0)\cdot\delta t}$.

Example

Let's use the above formulas in Matlab to compute the price of a (vanilla) European put with $S_0 = 45$, $K = 50$, $r = 0.1$, $\sigma = 0.4$ and maturity in two months, using $\delta t = 1/12$, i.e., a month. Do this by hand and build the lattices at the board.

Solution. Calculate in order $\delta t, u, d, p, e^{r \cdot \delta t}$ using the formulas on previous slides. Next, build a stock price binomial lattice starting from the left (initial time.) Finally, start on the right (final time) and work backwards to build the option price lattice, using the fact that we know the final option prices match the payoff curve. Compare the value of the option at time zero to what we obtain using the bseurput.m function.

Question: How would we modify this procedure to account for an American option? This gives us a method for computing the value of an American option!!

Example Calculations

Here's how we would calculate by hand:

```
>T = 2/12
>S0 = 45
>K = 50
>dt = 1/12
>r = 0.1
>sigma = 0.4
>u = exp(sigma*sqrt(dt))
>d = 1/u
>G = exp(r*dt)
>p = (G-d)/(u-d)
>S1 = S0*u, S2 = S0*d, S3 = S0*u*u, S4 = S0*u*d, S5 =
S0*d*d
>f3 = max(K-S3,0), f4 = max(K-S4,0), f5 = max(K-S5,0)
>f1 = (p*f3+(1-p)*f4)/G, f2 = (p*f4 + (1-p)*f5)/G
>f0 = (p*f1 + (1-p)*f2)/G
>bseurput(S0,K,r,T,0,sigma,0)
```

American Put Variation

Now what if the put option were American?
The difference: *We can exercise the option early.* In particular, we will **always** do this if it is needed to keep us above the payoff curve.
So how does this change the calculations? Discuss.

American Put Calculations:

The values of f3, f4 and f5 are ok as they stand. Cursor up to the lines defining f0, f1 and f2 and ensure that your price stays above the payoff curve.

- `>f1 = max(K-S1,(p*f3+(1-p)*f4)/G), f2 = max(K-S2,(p*f4 + (1-p)*f5)/G)`
  `>f0 = max(K-S0,(p*f1 + (1-p)*f2)/G)`
- Just for the record:
  `>[price,lattice] = LatticeAmPut(S0,K,r,T,sigma,2)`

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Dividend-Paying Stocks

In some situations on previous slide, closed formulas are out the window, and numerical methods must be used. An exception:

A simple model of dividend-paying stocks:

Assume dividends are paid continuously at a rate D_0. What changes in our model?

- Dividend payouts reduce the asset price, so the proper model here is $dS = \sigma S\, dW + (\mu - D_0)\, S\, dt$
- One can deduce a Black-Scholes equation for option price $f(S,t)$ of the form
$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D_0)\, S \frac{\partial f}{\partial S} - rf = 0.$$
- The Black-Scholes formulas carry over, so Matlab functions `bseurcall.m`, `bseurput.m` and `eurcallgreeks.m` work fine for stocks with continuous dividends.

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