JDEP 384H: Numerical Methods in Business

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110 Kaufmann Center

Outline

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1. Basic Financial Assets and Related Issues
2. BT 1.4: Derivatives
   - The Basics
   - Black-Scholes
   - American Options and Binomial Lattices

The Greeks

Tools for Financial Analysis:

- $\Delta = \dfrac{\partial f(S,t)}{\partial S}$: "delta" measures sensitivity of portfolio to small variations in the stock price (analogous to duration in bonds.)
- $\Theta = \dfrac{\partial f(S,t)}{\partial t}$: "theta" measures sensitivity of portfolio to small variations in time (useful as expiry nears.)
- $\Gamma = \dfrac{\partial^2 f(S,t)}{\partial S^2}$: "gamma" measures sensitivity of portfolio to smaller effects (analogous to convexity in bonds.)
- $\nu = \dfrac{\partial f(S,t)}{\partial \sigma}$: the "vega" measures sensitivity to volatility.
- $\rho = \dfrac{\partial f(S,t)}{\partial r}$: the "rho" measures sensitivity to change in interest rate.

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Simulating a Random Walk:

This is easy with Matlab. For example, the random walk $dS = \sigma S dX + \mu S dt$ $\mu = 0.07$ and $\sigma = 0.03$, $S(0) = 100$. Do the following Matlab commands.

```
>mu = 0.06
>sigma = 0.03
>s = zeros(53,1);
>s(1)=100;
>dt = 1/52;
>dx = sqrt(dt)*randn(52,1);
>for ii=1:52,s(ii+1) =
s(ii)+s(ii)*(sigma*dx(ii)+mu*dt);end
>plot(s), hold on, grid % now repeat experiment
```

European Call/Put Calculations

Matlab Tools:

The functions bseurcall, bseurput and eurcallgreeks are helpful.

- Get help on each.
- Make a plot of call prices for a range of stock values from 20 to 80 on a stock with a strike price of 50, risk-free rate of 5.5% and volatility $\sigma = 0.4$ in steps of one month, given expiry is six months from now.
- Make tables of the greeks at times 0 and 3 months of each graph above (try Δ, Θ, Γ and ν.)

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Critique of Black-Scholes Equations

Final remarks on European options:

1. Some of the Greeks are known analytically, e.g., Δ. Others have to be estimated numerically.
2. A larger question: how do we estimate the other parameters of the problem, namely, μ, σ. Even worse, what if they change with time (they do!)?
3. Are variations on the theme amenable to this methodology? For example, what about a dividend yielding stock? Or a cash or nothing call?
4. How do American options muddy the picture? How much different are they from European options?

Dividend-Paying Stocks

In some situations on previous slide, closed formulas are out the window, and numerical methods must be used. An exception:

A simple model of dividend-paying stocks:

Assume dividends are paid continuously at a rate D_0. What changes in our model?

- Dividend payouts reduce the asset price, so the proper model here is $dS = \sigma S\, dW + (\mu - D_0)\, S\, dt$
- One can deduce a Black-Scholes equation for option price $f(S, t)$ of the form
$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D_0)\, S \frac{\partial f}{\partial S} - rf = 0.$$
- The Black-Scholes formulas carry over, so Matlab functions `bseurcall.m`, `bseurput.m` and `eurcallgreeks.m` work fine for stocks with continuous dividends.

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American Options

The difference:

You may exercise your option early. Why bother?

- Look at the graphs of an European put with $r = 0.1$, $\sigma = 0.4$, $T = 1$, $t = 0.5$ and strike price $K = 1$, and the payoff curve on $[0, 3]$.
- Ditto for an European call.
- Ditto for an European call on a dividend-paying stock with with $D_0 = 0.06$.
- Help is on the next slide.

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Calculating Curves

O.K., let's execute this Matlab code:

```
>help bseurput
>S = 0:0.01:2;
>K = 1
>r = 0.1
>T = 1
>t = 0.5
>sigma = 0.4
>D0 = 0
>plot(S,bseurput(S,K,r,T,t,sigma,D0)
>hold on
>plot(S,max(K-S,0)) % the payoff curve for a put
```

The Problem with American Options

The Heart of the Difficulty:

- We enforce an inequality, e.g., $P(S,t) \geq \max\{K - S, 0\}$ that fights with the Black-Scholes equation.
- The new problem becomes what is called a linear complementarity problem: Solve

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \leq 0$$

  subject to the constraint $P(S,t) \geq \max\{K - S, 0\}$ where if one inequality is strict, then the other is an equality.
- There are no closed form solutions to these problems. They need advanced numerical methods of approximation.

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