JDEP 384H: Numerical Methods in Business

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Lecture 11, February 13, 2007
110 Kaufmann Center

Outline

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Value-at-Risk

Our last topic in this section is a brief look at measuring **short term** riskiness of a portfolio of stocks. The measure is a number called the **value-at-risk** (VaR) of the portfolio.

How much could we lose? We suppose that:

- Our portfolio's current value is W_0 and the future (random) wealth is $W = W_0 (1 + R)$ in a (short) time interval δt.
- So our change in wealth over this time is

$$\delta W = W - W_0 = W_0 R$$

- We ask: with a confidence level of $1 - \alpha$ what is the worse that could happen, i.e., the (positive) number VaR such that

$$P[\delta W \leq -VaR] = 1 - \alpha.$$

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How much could we lose?

- Last probability is same as $P\left[R \leq -\frac{VaR}{W_0}\right] = 1 - \alpha$.
- Now assume random rate of return R has a known distribution with c.d.f. $F(r)$, so that

$$P\left[R \leq -\frac{VaR}{W_0}\right] = F\left(\frac{VaR}{W_0}\right).$$

- We say that at a confidence level α (e.g., $\alpha = 0.95$), We look for the number VaR such that

$$F\left(r_{1-\alpha}\right) = F\left(-\frac{VaR}{W_0}\right) = 1 - \alpha$$

and solve for VaR.

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Text example (p. 85):

We have a portfolio of two assets with weights 2/3 and 1/3. Daily volatilities are $\sigma_1 = 2\%$ and $\sigma_2 = 1\%$, which translate into variances of $\sigma^2 \cdot \delta t$ for the time period, where σ is the daily volatility of the portfolio. Also, (daily) correlation is $\rho = 0.7$ and δt is 10 days. Assume the rate of return is normally distributed with mean zero and standard deviation given by the volatility of the portfolio. What is the VaR on a ten million dollar investment?

- Calculate the volatility of the portfolio.
- Use the `norm_inv.m` function to determine the VaR.
- What if the stocks were negatively correlated by $\rho = -.7$?
- We should factor in the drift μ, if the time line is very long. How? Add growth to VaR.
- VaR suffers some severe defects. For one, it is not subadditive, that is, we could have assets A and B such that $\text{VaR}(A+B) > \text{VaR}(A) + \text{VaR}(B)$. This is odd indeed!

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Matlab Analysis of Example

Portfolio of two assets with weights 2/3 and 1/3:

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```
>w = [2/3;1/3],W0=1e7
>s1 = 0.02, s2 = 0.01, r = 0.7
>Sperday = [s1^2, r*s1*s2; r*s1*s2, s2^2]
>deltat = 10
>S = deltat*Sperday
>sigma2 = w'*S*w
>alpha = 0.95
>ralpha = norm_inv(1-alpha,0,sigma2)
>VaR = -W0*ralpha % now try it at 99% confidence.
>VaR = -W0*ralpha - W0*0.2/365 %with 20% annual drift
```

Outline

The Basics

Review of Derivative:

A financial contract binding holders to certain financial rights or obligations, contingent on certain conditions which may involve the price of other securities.

- **Forward contract**: an agreement to buy or sell a risky asset at a specified future time T, called the **delivery date**, for a price F fixed at the present moment, called the **forward price**. Buyer is said to take a **long** position and seller a **short** position.
- **Call (put) option:** a contract giving the right to buy (sell) an underlying asset for a price K, called the **strike price**, at a certain date T, called the **expiration (maturity, expiry) date**. If exercise is allowed only at T, this is an European option. If at any time up to T, this is an American option.

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Discussion

Case of a call or put option with underlying asset a stock whose price we view as a random variable $S(t)$ that varies with time t:

- Key quantity for a long position is $S(T) - K$. If positive, exercise the option, then sell the stock immediately for a gain. Otherwise, don't exercise the option. In any case, your **payoff** is
$$\max\{S(T) - K, 0\}.$$
- If this is positive, you're said to be *in-the-money*.
- If zero you're *at-the-money*.
- If negative, *you're out-of-the-money*.
- A complementary situation holds for a put option, where your payoff is
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Pricing a Derivative

The key to getting a grip on pricing of derivatives is to keep the arbitrage principle in mind. Using it, one can verify

- A forward contract for delivery at time T of an asset with spot price $S(0)$ at time $t = 0$ (now) has fair price given by

  $$F = S(0)e^{rT},$$

  where r is the prevailing risk-free interest rate (with continuous compounding, for convenience.) Why so?
- If an asset has spot price $S(0)$, European call or put options on it with the same exercise price X at time $t = T$ and fair prices C or P, then these are related by the *call-put parity formula*

  $$C + Xe^{-rT} = P + S(0)$$

Call-Put Parity

$$C + Xe^{-r(T-t)} = P + S(0)$$

Let's verify this formula by considering a portfolio at time t before expiry time T consisting of

- one asset (you own it)
- a long put with strike price X (you own it at fair price P)
- a short call with strike price X (you borrowed it at fair price C)

Let V be the value of this portfolio. Calculate its value at times t (it's $V(t) = S(t) + P - C$) and at time T and ask: how much would I pay for this? Keep in mind that the time to expiry is now $T - t$.

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Differentials: Window to the Future

The Problem

We know the value of a function $f(t)$ at time t (perhaps a stock value, etc). We want to know it in the future – even a short time into future – say $f(t + dt)$ where dt is a small increment of time.

- A quantity we want is $\Delta f \equiv f(t + dt) - f(t)$.
- Differentials to the rescue

$$\Delta f = f(t + dt) - f(t) = \int_t^{t+dt} f'(s)\, ds \approx f(t)\, dt$$

- Why differentials? Because we don't know the future. And...

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Differentials: They Play Nice

Properties of differentials:

There are many. A small sampling:

- If $f(x) = a \cdot g(x) + b \cdot h(x)$, then $df = a \cdot dg + b \cdot dh$
- If $f(x) = g(x) \cdot h(x)$, then $df = g(x) \cdot dh + h(x) \cdot dg$
- If $f(x) = h(x)/g(x)$, then $df = (g(x)\,df - h(x) \cdot dg)/g(x)^2$
- Let $f = f(x,t)$ a function of two variables. We can take **partial derivatives** with respect to x and t separately by treating them as the only variables, yielding $\frac{\partial f}{\partial x}(x,t)$ and $\frac{\partial f}{\partial t}(x,t)$.
- The Chain Rule:

$$df = \frac{\partial f}{\partial x}(x,t)\,dx + \frac{\partial f}{\partial t}(x,t)\,dt$$

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1. Basic Financial Assets and Related Issues
   - Value-at-Risk
2. BT 1.4: Derivatives
   - The Basics
   - Black-Scholes

Onward to Black-Scholes

At this point, let's take some time and cruise through the stochastic processes section of our ProbStatLectures (at least up to the Stochastic Integrals section.) Highlights:

Risky asset price $S(t)$:

- Is a random variable for each time t.
- Is described as a random process $\frac{dS}{S} = \sigma\, dX + \mu\, dt$ where $\sigma\, dX$ is the risky part and $\mu\, dt$ is the risk-free part. Discuss volatility σ and drift μ.
- If $f(S,t)$ is the price of a call or put, Ito's Lemma tells us that

$$df = \sigma S \frac{\partial f}{\partial S} dX + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt.$$

Onward to Black-Scholes

At this point, let's take some time and cruise through the stochastic processes section of our ProbStatLectures (at least up to the Stochastic Integrals section.) Highlights:

Risky asset price $S(t)$:

- Is a random variable for each time t.
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The Equation

Here's the scenario: Consider a portfolio consisting of one option at a price of $f(S,t)$and Δ short shares of the corresponding stock at price S. So the value of the portfolio is

$$V = f(S,t) - \Delta \cdot S$$

Black-Scholes Derivation:

Analyze the differential of the price:

- $dV = df - \Delta \cdot dS$. So use Ito:
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- If the portfolio were risk-free, we would have $dV = Vr\, dt$
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