JDEP 384H: Numerical Methods in Business

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Department of Mathematics

Lecture 11, February 13, 2007
110 Kaufmann Center
Outline

1. Basic Financial Assets and Related Issues
   - Value-at-Risk

2. BT 1.4: Derivatives
   - The Basics
   - Black-Scholes
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Our last topic in this section is a brief look at measuring short term riskiness of a portfolio of stocks. The measure is a number called the value-at-risk (VaR) of the portfolio.

How much could we lose? We suppose that:

- Our portfolio’s current value is $W_0$ and the future (random) wealth is $W = W_0 (1 + R)$ in a (short) time interval $\delta t$.
- So our change in wealth over this time is
  \[
  \delta W = W - W_0 = W_0 R
  \]
- We ask: with a confidence level of $1 - \alpha$ what is the worse that could happen, i.e., the (positive) number VaR such that
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Value-at-Risk

How much could we lose?

- Last probability is same as \( P \left[ R \leq -\frac{VaR}{W_0} \right] = 1 - \alpha \).
- Now assume random rate of return \( R \) has a known distribution with c.d.f. \( F(r) \), so that
  \[
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- We say that at a confidence level \( \alpha \) (e.g., \( \alpha = 0.95 \)), We look for the number \( VaR \) such that
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  F \left( r_{1-\alpha} \right) = F \left( -\frac{VaR}{W_0} \right) = 1 - \alpha
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  and solve for \( VaR \).
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We have a portfolio of two assets with weights $\frac{2}{3}$ and $\frac{1}{3}$. Daily volatilities are $\sigma_1 = 2\%$ and $\sigma_2 = 1\%$, which translate into variances of $\sigma^2 \cdot \delta t$ for the time period, where $\sigma$ is the daily volatility of the portfolio. Also, (daily) correlation is $\rho = 0.7$ and $\delta t$ is 10 days. Assume the rate of return is normally distributed with mean zero and standard deviation given by the volatility of the portfolio. What is the VaR on a ten million dollar investment?

- Calculate the volatility of the portfolio.
- Use the `norm_inv.m` function to determine the VaR.
- What if the stocks were negatively correlated by $\rho = -0.7$?
- We should factor in the drift $\mu$, if the time line is very long. How? Add growth to VaR.
- VaR suffers some severe defects. For one, it is not subadditive, that is, we could have assets $A$ and $B$ such that $\text{VaR} (A + B) > \text{VaR} (A) + \text{VaR} (B)$. This is odd indeed!
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Matlab Analysis of Example

Portfolios of two assets with weights 2/3 and 1/3:

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```matlab
> w = [2/3; 1/3], W0 = 1e7
> s1 = 0.02, s2 = 0.01, r = 0.7
> Sperday = [s1^2, r*s1*s2; r*s1*s2, s2^2]
> deltat = 10
> S = deltat*Sperday
> sigma2 = w'*S*w
> alpha = 0.95
> ralpha = norm_inv(1-alpha, 0, sigma2)
> VaR = -W0*ralpha % now try it at 99% confidence.
> VaR = -W0*ralpha - W0*0.2/365 % with 20% annual drift
```
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Review of Derivative:

A financial contract binding holders to certain financial rights or obligations, contingent on certain conditions which may involve the price of other securities.

- **Forward contract**: an agreement to buy or sell a risky asset at a specified future time $T$, called the delivery date, for a price $F$ fixed at the present moment, called the forward price. Buyer is said to take a long position and seller a short position.

- **Call (put) option**: a contract giving the right to buy (sell) an underlying asset for a price $K$, called the strike price, at a certain date $T$, called the expiration (maturity, expiry) date. If exercise is allowed only at $T$, this is an European option. If at any time up to $T$, this is an American option.
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Case of a call or put option with underlying asset a stock whose price we view as a random variable \( S(t) \) that varies with time \( t \):

- Key quantity for a long position is \( S(T) - K \). If positive, exercise the option, then sell the stock immediately for a gain. Otherwise, don’t exercise the option. In any case, your payoff is
  \[
  \max \{ S(T) - K, 0 \}. 
  \]
  - If this is positive, you’re said to be *in-the-money*.
  - If zero you’re *at-the-money*.
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A complementary situation holds for a put option, where your payoff is

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Pricing a Derivative

The key to getting a grip on pricing of derivatives is to keep the arbitrage principle in mind. Using it, one can verify

1. A forward contract for delivery at time $T$ of an asset with spot price $S(0)$ at time $t = 0$ (now) has fair price given by

   $$F = S(0)e^{rT},$$

   where $r$ is the prevailing risk-free interest rate (with continuous compounding, for convenience.) Why so?

2. If an asset has spot price $S(0)$, European call or put options on it with the same exercise price $X$ at time $t = T$ and fair prices $C$ or $P$, then these are related by the call-put parity formula

   $$C + Xe^{-rT} = P + S(0)$$
Call-Put Parity

\[ C + Xe^{-r(T-t)} = P + S(0) \]

Let's verify this formula by considering a portfolio at time \( t \) before expiry time \( T \) consisting of

- one asset (you own it)
- a long put with strike price \( X \) (you own it at fair price \( P \))
- a short call with strike price \( X \) (you borrowed it at fair price \( C \))

Let \( V \) be the value of this portfolio. Calculate its value at times \( t \) (it’s \( V(t) = S(t) + P - C \)) and at time \( T \) and ask: how much would I pay for this? Keep in mind that the time to expiry is now \( T - t \).
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The Problem

We know the value of a function $f(t)$ at time $t$ (perhaps a stock value, etc). We want to know it in the future – even a short time into future – say $f(t + dt)$ where $dt$ is a small increment of time.

- A quantity we want is $\Delta f \equiv f(t + dt) - f(t)$.
- Differentials to the rescue

$$\Delta f = f(t + dt) - f(t) = \int_{t}^{t+dt} f'(s) \, ds \approx f(t) \, dt$$

- Why differentials? Because we don’t know the future. And...
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Differentials: Window to the Future

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Differentials: They Play Nice

Properties of differentials:

There are many. A small sampling:

- If \( f(x) = a \cdot g(x) + b \cdot h(x) \), then \( df = a \cdot dg + b \cdot dh \)
- If \( f(x) = g(x) \cdot h(x) \), then \( df = g(x) \cdot dh + h(x) \cdot dg \)
- If \( f(x) = h(x)/g(x) \), then
  \[
  df = \frac{g(x) \, df - h(x) \cdot dg}{g(x)^2}
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- Let \( f = f(x, t) \) a function of two variables. We can take partial derivatives with respect to \( x \) and \( t \) separately by treating them as the only variables, yielding \( \frac{\partial f}{\partial x} (x, t) \) and \( \frac{\partial f}{\partial t} (x, t) \).
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  df = \frac{g(x) \cdot df - h(x) \cdot dg}{g(x)^2}
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- Let \( f = f(x, t) \) a function of two variables. We can take partial derivatives with respect to \( x \) and \( t \) separately by treating them as the only variables, yielding \( \frac{\partial f}{\partial x} (x, t) \) and \( \frac{\partial f}{\partial t} (x, t) \).
- The Chain Rule:

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df = \frac{\partial f}{\partial x} (x, t) \, dx + \frac{\partial f}{\partial t} (x, t) \, dt
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Properties of differentials:

There are many. A small sampling:

- If \( f(x) = a \cdot g(x) + b \cdot h(x) \), then \( df = a \cdot dg + b \cdot dh \)
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Outline

1. Basic Financial Assets and Related Issues
   - Value-at-Risk

2. BT 1.4: Derivatives
   - The Basics
   - Black-Scholes
Onward to Black-Scholes

At this point, let’s take some time and cruise through the stochastic processes section of our ProbStatLectures (at least up to the Stochastic Integrals section.) Highlights:

Risky asset price $S(t)$:

- Is a random variable for each time $t$.
- Is described as a random process $\frac{dS}{S} = \sigma \, dX + \mu \, dt$ where $\sigma \, dX$ is the risky part and $\mu \, dt$ is the risk-free part. Discuss volatility $\sigma$ and drift $\mu$.
- If $f(S, t)$ is the price of a call or put, Ito’s Lemma tells us that

$$df = \sigma S \frac{\partial f}{\partial S} \, dX + \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) \, dt.$$
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**Black-Scholes Derivation:**

Analyze the differential of the price:

- $dV = df - \Delta \cdot dS$. So use Ito:
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- If the portfolio were risk-free, we would have $dV = Vr dt$
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