JDEP 384H: Numerical Methods in Business

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110 Kaufmann Center

Outline

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Mean-Variance Portfolio Optimization

Following text, we'll stick to a portfolio of two risky assets for purpose of illustration. Rather than use absolute quantities, we use rates of return. For simplicity, examine a portfolio of two assets.

- The absolute x_i above are replaced by fractions w_i, where $w_1 + w_2 = 1$ and we denote the vector $[w_1; w_2]$ by $\mathbf{w}$.
- Assume no short positions, so the $w_i \geq 0$.
- Rates of return on investments are r_1, r_2, respectively, so rate of return of portfolio is $r = w_1 r_1 + w_2 r_2$.
- The expected returns are $\bar{r}_1$, $\bar{r}_2$ and $\bar{r} = w_1 \bar{r}_1 + w_2 \bar{r}_2$.
- The r.v.'s r_1, r_2 have covariance matrix Σ, so that the variance of our portfolio is

$$\mathrm{Var}\left(w_1 r_1 + w_2 r_2\right) = \mathbf{w}^T \Sigma \mathbf{w}.$$

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Mean-Variance Portfolio Optimization

Problems:

- For a given expected return $\bar{r}_T$, what weighting gives the minimum variance?
- Answer: the solution to the quadratic programming problem:

$$\min \mathbf{w}^T \Sigma \mathbf{w}$$

subject to

$$\begin{aligned} \mathbf{w}^T \bar{\mathbf{r}} &= \bar{r}_T \\ \mathbf{w}^T \mathbf{1} &= 1 \end{aligned}$$

- What is the range of possible expected returns? Examine the definition.

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Mean-Variance Portfolio Optimization

Definition

A portfolio is **efficient** if it is not possible to obtain a higher expected return without increasing the risk.

Definition

An **efficient frontier** is a graph of efficient portfolio's risk versus expected return.

Example

From text, p. 74, suppose two assets have expected earning rates $\bar{r}_1 = 0.2$, $\bar{r}_2 = 0.1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.4$ and $\sigma_{12} = -0.1$. Design an efficient frontier for this problem using Matlab. How would we find the leftmost point on the graph?

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Example Analysis

To solve this problem:

- Write out the system to be solved in detail.
- Notice anything about the weight constraints?
- So use Matlab as follows:

```
>ereturn = (0.1:.01:0.2)
>rhs = [ones(size(ereturn));ereturn]
>coef = [1 1 ; 0.2 0.1]
>wts = coef\rhs
>covar = [0.2 -0.1; -0.1 0.4]
>risks2 = wts'*covar*wts
>risks2 = diag(risks2)
>risks = sqrt(risks2)
>plot(risks,ereturn),grid
>xlabel('Risk (Standard Deviation)')
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So where is the efficient frontier? (Try using `min` on `risks`)

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Example Analysis

Finding Boundary of Efficient Frontier:

The left-most point on the efficient frontier corresponds to solution of the quadratic programming program

$$\min \mathbf{w}^T \Sigma \mathbf{w}$$

subject to $\mathbf{w}^T \mathbf{1} = 1$ and $\mathbf{w} \geq 0$. In our example, this problem can be reduced to one variable and solved by hand (substitute $1 - w_1$ for w_2.) This is the **minimum risk portfolio**.

Putting It All Together

Now we know how to optimize a portfolio at a given return level. But this can't answer the question: what risk level is acceptable? We need to return to the utility function.

Example

Discuss the kind of problem that has to be solved if the investor's utility function is modeled by

$$u = \bar{r} - 0.005 \cdot A \cdot \sigma^2$$

where A is linked to the investor's risk aversion, say with a typical range of 2 to 4, in the two asset case considered above. Test for the optimal portfolio *assuming* that it occurs along the efficient frontier.

Putting It All Together

Start with the observation that if we use a weighted blend of the stocks then

$$u(\mathbf{w}) = [0.2, 0.1]\,\mathbf{w} - 0.005 \cdot A \cdot \mathbf{w}^T \Sigma \mathbf{w}$$

- Find the expected returns for each weight, $A = 3, 60$:
```
>u = [0.2 0.1]*wts - 0.005*3*risks'
>plot(wts(1,:),u)
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- How would we handle this problem if we allowed short positions? Is there even a solution?
- Are there limits on short positions? How would we formulate this in a quadratic programming context?

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Value-at-Risk

Our last topic in this section is a brief look at measuring **short term** riskiness of a portfolio of stocks. The measure is a number called the **value-at-risk** (VaR) of the portfolio.

How much could we lose? We suppose that:

- Our portfolio's current value is W_0 and the future (random) wealth is $W = W_0(1 + R)$ in a (short) time interval δt.
- So our change in wealth over this time is

$$\delta W = W - W_0 = W_0 R$$

- We ask: with a confidence level of $1 - \alpha$ what is the worse that could happen, i.e., the (positive) number VaR such that

$$P[\delta W \leq -VaR] = 1 - \alpha.$$

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- Last probability is same as $P\left[R \leq -\frac{VaR}{W_0}\right] = 1 - \alpha$.
- Now assume random rate of return R has a known distribution with c.d.f. $F(r)$, so that

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- We say that at a confidence level α (e.g., $\alpha = 0.95$), We look for the number VaR such that

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Example

Text example (p. 85):

We have a portfolio of two assets with weights 2/3 and 1/3. Daily volatilities are $\sigma_1 = 2\%$ and $\sigma_2 = 1\%$, which translate into variances of $\sigma^2 \cdot \delta t$ for the time period, where σ is the daily volatility of the portfolio. Also, (daily) correlation is $\rho = 0.7$ and δt is 10 days. Assume the rate of return is normally distributed with mean zero and standard deviation given by the volatility of the portfolio. What is the VaR on a ten million dollar investment?

- Calculate the volatility of the portfolio.
- Use the `norm_inv.m` function to determine the VaR.
- What if the stocks were negatively correlated by $\rho = -.7$?
- We should factor in the drift μ, if the time line is very long. How? Add growth to VaR.
- VaR suffers some severe defects. For one, it is not subadditive, that is, we could have assets A and B such that $\mathrm{VaR}(A+B) > \mathrm{VaR}(A) + \mathrm{VaR}(B)$. This is odd indeed!

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