Outline

1. Basic Financial Assets and Related Issues
   - Mean-Variance Portfolio Optimization
   - Value-at-Risk
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Following text, we’ll stick to a portfolio of two risky assets for purpose of illustration. Rather than use absolute quantities, we use rates of return. For simplicity, examine a portfolio of two assets.

- The absolute $x_i$ above are replaced by fractions $w_i$, where $w_1 + w_2 = 1$ and we denote the vector $[w_1; w_2]$ by $w$.
- Assume no short positions, so the $w_i \geq 0$.
- Rates of return on investments are $r_1$, $r_2$, respectively, so rate of return of portfolio is $r = w_1 r_1 + w_2 r_2$.
- The expected returns are $\bar{r}_1$, $\bar{r}_2$ and $\bar{r} = w_1 \bar{r}_1 + w_2 \bar{r}_2$.
- The r.v.’s $r_1$, $r_2$ have covariance matrix $\Sigma$, so that the variance of our portfolio is

$$\text{Var} (w_1 r_1 + w_2 r_2) = w^T \Sigma w.$$
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Mean-Variance Portfolio Optimization

Problems:

- For a given expected return \( \bar{r}_T \), what weighting gives the minimum variance?
- Answer: the solution to the quadratic programming problem:

\[
\min w^T \Sigma w
\]

subject to

\[
w^T \bar{r} = \bar{r}_T
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w^T 1 = 1
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**Definition**

A portfolio is **efficient** if it is not possible to obtain a higher expected return without increasing the risk.

**Definition**

An efficient frontier is a graph of efficient portfolio’s risk versus expected return.

**Example**

From text, p. 74, suppose two assets have expected earning rates $\bar{r}_1 = 0.2$, $\bar{r}_2 = 0.1$, $\sigma^2_1 = 0.2$, $\sigma^2_2 = 0.4$ and $\sigma_{12} = -0.1$. Design an efficient frontier for this problem using Matlab. How would we find the leftmost point on the graph?
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Example Analysis

To solve this problem:

- Write out the system to be solved in detail.
- Notice anything about the weight constraints?
- So use Matlab as follows:
  ```matlab
ereturn = (0.1:.01:0.2)
rhs = [ones(size(ereturn));ereturn]
coef = [1 1; 0.2 0.1]
wts = coef\rhs
covar = [0.2 -0.1; -0.1 0.4]
risks2 = wts’*covar*wts
risks2 = diag(risks2)
risks = sqrt(risks2)
plot(risks,ereturn),grid
xlabel(’Risk (Standard Deviation)’)   
ylabel(’Expected Return’)
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So where is the efficient frontier? (Try using min on risks)
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So where is the efficient frontier? (Try using `min` on `risks`)
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> plot(risks, ereturn), grid
> xlabel(’Risk (Standard Deviation)’)
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So where is the efficient frontier? (Try using min on risks)
Finding Boundary of Efficient Frontier:

The left-most point on the efficient frontier corresponds to solution of the quadratic programming program

$$\min w^T \Sigma w$$

subject to $w^T 1 = 1$ and $w \geq 0$. In our example, this problem can be reduced to one variable and solved by hand (substitute $1 - w_1$ for $w_2$.) This is the *minimum risk portfolio*. 
Now we know how to optimize a portfolio at a given return level. But this can’t answer the question: what risk level is acceptable? We need to return to the utility function.

**Example**

Discuss the kind of problem that has to be solved if the investor’s utility function is modeled by

\[ u = \bar{r} - 0.005 \cdot A \cdot \sigma^2 \]

where \( A \) is linked to the investor’s risk aversion, say with a typical range of 2 to 4, in the two asset case considered above. Test for the optimal portfolio *assuming* that it occurs along the efficient frontier.
Start with the observation that if we use a weighted blend of the stocks then

\[ u(w) = [0.2, 0.1]w - 0.005 \cdot A \cdot w^T \Sigma w \]

- Find the expected returns for each weight, \( A = 3, 60 \):
  \[
  > u = [0.2 \ 0.1] \ast \text{wts} - 0.005 \ast 3 \ast \text{risks'}
  > \text{plot(wts}(1,:),u)
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- How would we handle this problem if we allowed short positions? Is there even a solution?
- Are there limits on short positions? How would we formulate this in a quadratic programming context?
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Our last topic in this section is a brief look at measuring short term riskiness of a portfolio of stocks. The measure is a number called the value-at-risk (VaR) of the portfolio.

How much could we lose? We suppose that:

- Our portfolio’s current value is $W_0$ and the future (random) wealth is $W = W_0 (1 + R)$ in a (short) time interval $\delta t$.
- So our change in wealth over this time is
  \[
  \delta W = W - W_0 = W_0 R
  \]
- We ask: with a confidence level of $1 - \alpha$ what is the worse that could happen, i.e., the (positive) number VaR such that
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  P[\delta W \leq -VaR] = 1 - \alpha.
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How much could we lose?

- Last probability is same as $P \left[ R \leq - \frac{VaR}{W_0} \right] = 1 - \alpha$.

- Now assume random rate of return $R$ has a known distribution with c.d.f. $F(r)$, so that

$$P \left[ R \leq - \frac{VaR}{W_0} \right] = F \left( \frac{VaR}{W_0} \right).$$

- We say that at a confidence level $\alpha$ (e.g., $\alpha = 0.95$), we look for the number $VaR$ such that

$$F (r_{1-\alpha}) = F \left( - \frac{VaR}{W_0} \right) = 1 - \alpha$$

and solve for $VaR$. 
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We have a portfolio of two assets with weights 2/3 and 1/3. Daily volatilities are $\sigma_1 = 2\%$ and $\sigma_2 = 1\%$, which translate into variances of $\sigma^2 \cdot \delta t$ for the time period, where $\sigma$ is the daily volatility of the portfolio. Also, (daily) correlation is $\rho = 0.7$ and $\delta t$ is 10 days. Assume the rate of return is normally distributed with mean zero and standard deviation given by the volatility of the portfolio. What is the VaR on a ten million dollar investment?

- Calculate the volatility of the portfolio.
- Use the `norm_inv.m` function to determine the VaR.
- What if the stocks were negatively correlated by $\rho = -0.7$?
- We should factor in the drift $\mu$, if the time line is very long. How? Add growth to VaR.
- VaR suffers some severe defects. For one, it is not subadditive, that is, we could have assets $A$ and $B$ such that $\text{VaR}(A + B) > \text{VaR}(A) + \text{VaR}(B)$. This is odd indeed!
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Text example (p. 85):

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