JDEP 384H: Numerical Methods in Business

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Lecture 10, February 8, 2007
110 Kaufmann Center
Outline

1. Basic Financial Assets and Related Issues
   - Bond Portfolio Immunization (Revisited)

2. BT 2.4: Portfolio Optimization
   - Utility Theory
   - Mean-Variance Portfolio Optimization
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Example

Use Matlab to determine the correct weighting of three bonds with durations 2, 4, 6 and convexities 12, 15, 20, respectively, if we are to shape a portfolio with duration 3 and convexity 13.

Solution. Use the First Pass strategy. Work this system out with Matlab. What happens if no short positions are allowed? What about the Second Pass?

With 3 bonds, we’re stuck. But increase the number by, say one, to 4 bonds. Now we have a new situation of 3 equations in 4 unknowns. Since unknowns exceed equations, we can expect infinitely many solutions if any at all (see Linear Algebra Lecture)! So which do we select?
Immunization Strategies

Idea: Use the extra degree(s) of freedom to turn the problem into a linear programming problem. For example, maximize the weighted yield of the portfolio. Say the bonds have yields $Y_1, Y_2, Y_3, Y_4$. The problem becomes: Maximize the objective function

$$f(w_1, w_2, w_3, w_4) = Y_1 w_1 + Y_2 w_2 + Y_3 w_3 + Y_3 w_3 + Y_4 w_4$$

subject the the three constraints as in first pass with four variables.

An example portfolio consists of:

a weighted combination (no short positions) of four bonds with durations 2, 3, 4, 6, convexities 12, 12.5, 15, 20, and yields 0.06, 0.061, 0.065, 0.07, respectively. How to maximize yield?

- Use Matlab to solve this problem. If there is a solution, what is the maximum yield?
- What if we relax the convexity constraint to having convexity at least 13? Portfolio yield?
- What if we drop the convexity constraint altogether? Portfolio yield and convexity?
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Quantifying Risk and Risk Tolerance

We examine portfolios of risky securities, such as stocks. Note: In fact, bonds have an element of risk too!

Risk:

How can we measure risk?

- The return on our investment is wealth $X$, which is now a random variable. So are the returns $R_i$ of each stock in our portfolio.
- As such, returns have an expected value (mean) $x = E[X]$ which is the weighted sum of the expected returns $r_i$ of the $i$th stock.
- Variability of a r.v. is measured by its standard deviation. Hence the risk of the $i$th stock is just $\sigma_i = \sqrt{\text{Var}(R_i)}$.
- Competing goals: maximize return, minimize risk.
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One way to quantify an investor’s preferences:

**Utility Function \( u(x) \) of payoff \( x \):**

A numerical measure of satisfaction gained from a payoff \( x \).

- Normally, \( u \) is monotone increasing with \( x \).
- Normally, \( u \) is concave (concave down) implies that \( u''(x) < 0 \), which implies that \( u'(x) \) is decreasing.
- Hence concavity is a measure of risk aversion, because it implies that each increment to wealth conveys progressively smaller increments to utility.
- Two examples: \( u(x) = \log x \) and \( u(x) = x - \frac{\lambda}{2}x^2 \) (\( x \leq 1/\lambda \)).

There are other types of utility functions, e.g., we could have \( u \) depend on the expected rate of return and variance

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u = r - 0.005 \cdot A \cdot \sigma^2
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Consequences of Utility Approach:

- Obtain measures of degree of risk aversion (Arrow-Pratt absolute and relative risk aversion coefficients):
  \[ R_u^a(x) = -\frac{u''(x)}{u'(x)} \quad \text{and} \quad R_u^r(x) = -\frac{u''(x)}{u'(x)} x \]

- Portfolio optimization becomes a math problem: Given initial wealth \( W_0 \), a set of assets with return \( R_i \) (a random variable), and portfolio with \( x_i \) dollars of \( i \)th asset,

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  \max E \left[ u\left( x_1 R_1 + \cdots + x_n R_n \right) \right] 
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  subject to \( x_1 + \cdots + x_n = W_0 \).
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Following text, we’ll stick to a portfolio of two risky assets for purpose of illustration. Rather than use absolute quantities, we use rates of return. For simplicity, examine a portfolio of two assets.

- The absolute $x_i$ above are replaced by fractions $w_i$, where $w_1 + w_2 = 1$ and we denote the vector $[w_1; w_2]$ by $w$.
- Assume no short positions, so the $w_i \geq 0$.
- Rates of return on investments are $r_1$, $r_2$, respectively, so rate of return of portfolio is $r = w_1 r_1 + w_2 r_2$.
- The expected returns are $\bar{r}_1$, $\bar{r}_2$ and $\bar{r} = w_1 \bar{r}_1 + w_2 \bar{r}_2$.
- The r.v.’s $r_1$, $r_2$ have covariance matrix $\Sigma$, so that the variance of our portfolio is

$$\text{Var} (w_1 r_1 + w_2 r_2) = w^T \Sigma w.$$
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Answer: the solution to the quadratic programming problem:

$$\min_w w^T \Sigma w$$

subject to

$$w^T \bar{r} = \bar{r}_T$$

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What is the range of possible expected returns? Examine the definition.
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A portfolio is **efficient** if it is not possible to obtain a higher expected return without increasing the risk.

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An **efficient frontier** is a graph of efficient portfolio’s risk versus expected return.

**Example**

From text, p. 74, suppose two assets have expected earning rates $\bar{r}_1 = 0.2$, $\bar{r}_2 = 0.1$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.4$ and $\sigma_{12} = -0.1$. Design an efficient frontier for this problem using Matlab. How would we find the leftmost point on the graph?
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