# JDEP 384H: Numerical Methods in Business

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110 Kaufmann Center

# Outline

1. Parameter Estimation

2. BT 2.1-2: Uncertainty and Basic Financial Assets
   - Modeling Uncertainty in Time
   - Basic Financial Assets

# Reminder of Rational File Management

## File Management in 384H:

We want to keep perfectly synchronized with all the latest course files. Here's how:

- Create for once and for all a JDEP384hS06 directory somewhere in your own home directory.
- Create subdirectories in which you'll put work, etc., but avoid the names of directories already existing in the course Public directory of the above name.
- Before, or at the start of class, grab a copy of ZipDir.zip which will be located in the current Weekx. Save it in the root of your JDEP384hS06 directory and do NOT unzip it.
- Go into the ZipDir.zip and copy all files inside directory ZipDir.
- Move to root of DEP384hS06 and paste all. Delete ZipDir.zip.

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# Generating Data

After examining the ProbStatLecture section on parameter estimation:
Let's generate data simulating the weighing of an object of true weight 10. Assume the experiment is performed 16 times in independent trials (take note: this is a rather small sample.) We'll assume the error has a $N(10, .0.02)$ distribution.

```
n = 16
mu = 10
sigma = sqrt(0.02)
randn('state',0)
data = sigma*randn(n,1) + mu;
```

We'll use this for the following experiments. Just for fun, let's histogram the random numbers:

```
hist(data);
x = min(data):.1:max(data);
hist(data,x);
```

# Estimation of Mean, Known Variance

**Notation:** Given a r.v. $X$ and probability $\gamma$, $x_\gamma$ is the number such that $F_X(x_\gamma) = \gamma$.
Since a c.d.f. is always monotone increasing, we expect that there always is such a number $x_\gamma$, provided that $F_X$ is continuous. In fact

$$x_\gamma = F_X^{-1}(\gamma).$$

# Estimation of Mean, Known Variance

**Key Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

has a standard normal distribution.

# Estimation of Mean, Known Variance

A calculation based on this fact shows:

**Confidence interval for $\mu$, $\sigma$ known, confidence coefficient $1-\alpha$:**

$$\overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Here we are using the fact that the standard normal distribution is symmetric about the origin, so that $z_{1-\alpha/2} = -z_{\alpha/2}$. Now use this fact on our data to construct a confidence interval for the true weight of the object. The function `stdn_inv` is helpful here.

# Example

We use the sample data from the previous slide and a confidence level of 95%. Now perform these calculations, and let's toss in a graph as well.

```
alpha = 0.05
za = stdn_inv(alpha/2)
meanx = mean(data)
c = meanx + za*sigma/sqrt(n)
d = meanx - za*sigma/sqrt(n)
xnodes = -5:.01:5;
plot(xnodes,stdn_pdf(xnodes));
hold on, grid;
y = 0:.01:stdn_pdf(za);
plot(za*ones(size(y)),y);
plot(-za*ones(size(y)),y);
```

# Estimation of Mean, Unknown Variance

**Sampling Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a Student's t distribution with $n-1$ degrees of freedom.

# Estimation of Mean, Unknown Variance

A calculation based on this fact shows:

**Confidence interval for $\mu$, $\sigma$ known, confidence coefficient $1-\alpha$:**

$$\overline{x} + t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{x} - t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $T$ is the Student's t distribution with $n-1$ degrees of freedom.

Here we are using the fact that the t distribution is symmetric about the origin, so that $t_{1-\alpha/2} = -t_{\alpha/2}$. Now use this fact on our data to construct a confidence interval for the true weight of the object. The function `tdis_inv` is helpful here.

# Estimation of Variance

**Key Theorem:**

Let $X_1, X_2, \ldots, X_n$ be i.i.d. normal r.v.'s with mean $\mu$ and variance $\sigma^2$. Then the statistic

$$Y = (n-1)\frac{S^2}{\sigma^2}$$

has a chi-square distribution with $n-1$ degrees of freedom.

# Estimation of Variance

A calculation based on this fact shows:

**Confidence interval for $\sigma$ , confidence coefficient $1 - \alpha$:**

$$\frac{(n-1)\, s^2}{\chi^2_{1-\alpha/2}} < \sigma^2 < \frac{(n-1)\, s^2}{\chi^2_{\alpha/2}}$$

where $\chi^2$ is the chi-square distribution with $n-1$ degrees of freedom.

The chi-square distribution is NOT symmetric, whence the form above. Now use this fact on our data to construct a confidence interval for the true weight of the object. The functions `chis_inv` and `chis_pdf` are helpful here.

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# Discrete Time: A Simple Binomial Model

**Binomial Model:**

assumes that changes in the value of a security in discrete time steps to two discrete values, one up and one down.

**Standard assumptions:**

1. Amount of growth/loss is governed by constant rates $d$ and $u$, called multiplicative shock rates $0 < d < u$, so that if $S_0$ is the initial value, then the value one time step later is either $uS_0$ or $dS_0$.
2. Probabilities of up or down values are $p_u$ and $p_v$ with $p_u + p_v = 1$.
3. Recombining the basic lattice enables one to model the state of the asset at an arbitrary number of subsequent time steps.

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## Differential Models:

assume that time over which asset changes is continuous, so asset price is $S(t)$.

- Without uncertainty, might have something like

$$\frac{dS}{dt} = rS,$$

  the model of continuous compounding at rate $r$.

- In differential form

$$dS = rS\,dt,$$

  which is a good approximation to the discrete model

$$\Delta S = rS\,\Delta t$$

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# Definitions

**Fixed-Income Security :**

a contract, purchased for a price $P$, that guarantees a specified cash flow stream (sequence of cash payments) over a specified interval of time. As a matter of convenience, we'll always measure cash in dollars.

**Standard assumptions:**

1. Cash flow stream is paid out at the end of constant time intervals, the *period* of the security. *Maturity* is the time over which the security extends in years or number of periods.
2. There is a prevailing risk-free interest annual rate $r$ at which one can borrow or lend money.
3. No-arbitrage principle: there is no admissible portfolio of funds with value zero initially and positive value one period later, with positive probability (no free lunch).

# Examples

**Example 1.** (simplest) zero coupon bond with period and maturity of one year. This security has a purchase price $P$ and face value (a.k.a. par value) $F$, which is paid at maturity. It has a *return* $R = F/P$ and rate of return $r = R - 1$ ($100r$ as a percent) so that

$$F = (1 + r) P.$$

**Example 2.** (not so simple): coupon bond with price $P$, face (or par) value $F$, coupon rate $r$ (remember to scale the rate appropriately if it is given as an annual percentage and the period is not a year) and maturity of $n$ periods. At the end of each period, a payment (coupon) of $rF$ is made and additionally, at maturity (end of last period) the face value $F$ is added to the payment.

# Stocks

**Stock:**

a contract, purchased at a price $S$ that entitles the owner to a (limited liability) share of the issuing firm.

**More Standard assumptions:**

1. *Randomness*: the future price of a stock $S(t)$ is a random variable, with at least two different values.
2. Stock prices are *positive*, *divisible* (one can hold a fraction or a share or bond), and *liquid* (stocks and bonds can be bought on demand at market price in arbitrary amounts.)
3. Both long and short positions in an asset are possible. Long position means that a portfolio has a positive number of shares, and short means it owes a positive number of shares. Short positions in risk-free securities may involve issuing and selling bonds. Short positions in a stock means the investor borrows the stock, sells it and uses the proceeds in some other way. Assume investors are always able to close a short position, i.e., repurchase the stock and return it to the owner.

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# Derivatives

**Derivative:**

A financial contract binding holders to certain financial rights or obligations, contingent on certain conditions which may involve the price of other securities.

- **Forward contract**: an agreement to buy or sell a risky asset at a specified future time $T$, called the **delivery date**, for a price $F$ fixed at the present moment, called the **forward price**. Buyer is said to take a **long** position and seller a **short** position.
- **Call (put) option:** a contract giving the right to buy (sell) an underlying asset for a price $K$, called the **strike price**, at a certain date $T$, called the **expiration (maturity, expiry) date**. If exercise is allowed only at $T$, this is an European option. If at any time up to $T$, this is an American option.

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# Discussion

Case of a call or put option with underlying asset a stock whose price we view as a random variable $S(t)$ that varies with time $t$:

- Key quantity for a long position is $S(T) - K$. If positive, exercise the option, then sell the stock immediately for a gain. Otherwise, don't exercise the option. In any case, your **payoff** is
$$\max\{S(T) - K, 0\}.$$
- If this is positive, you're said to be *in-the-money*.
- If zero you're *at-the-money*.
- If negative, *you're out-of-the-money*.

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# Discussion

A complementary situation holds for a put option, where your payoff is

$$\max\{K - S(T), 0\}.$$

Consider the graphs of a long position in a forward contract, call option or put option, where we plot the payoff curve against stock price at strike time (**expiry**.) We'll draw them at the board.

# Why Options?

Differences between plain investing and option investing can be spectacular:

- You own a single stock whose price at $t = 0$ is $S(0) = \$50$, then realize a gain thanks to $S(T) = \$55$. You've made a profit of
$$(55 - 50)/50 = 10\%.$$
- You own a call option whose value at $t = 0$ is \$10 and strike price is $X = \$50$. At maturity $T$ you make a profit of
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