JDEP 384H: Numerical Methods in Business

Instructor: Thomas Shores
Department of Mathematics

Lecture 25, April 17, 2007
110 Kaufmann Center
Outline

1. NT: Decision Analysis and Game Theory
   - An Intelligent Opponent: Game Theory
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   - An Intelligent Opponent: Game Theory
A Model Problem

The Problem:
Dominant strategy elimination and the more general maximin/minimax strategies will not solve the following problem.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>1 2 3</td>
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</tr>
<tr>
<td>1</td>
<td>2 3 -2</td>
<td>-1 4 0</td>
</tr>
<tr>
<td>2</td>
<td>-1 4 0</td>
<td>3 -2 -1</td>
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Solution:

Use *mixed strategies* instead of pure strategies, i.e., a probability vector \((x_1, x_2, x_3)\) for player 1 \((y = (y_1, y_2, y_3)\) for player 2) that maximizes for player 1 (minimizes for player 2) the payoff for all possible plays by the opponent.

- If the payoff table is converted into a matrix \(A\) \((m \times n\) in general), then the payoff for any pair of mixed strategies is

\[
p = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = x^T A y
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- Player 1’s goal: Find probability vector \(x\) solving

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\max_x \min_y x^T A y
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Linear Programming to the Rescue:

We can solve either problem of the previous frame with linear programming tools as follows.

- **Key fact**: Both problems have a solution with common payoff $p$. In fact, they solve the linear programming programs $\max p \text{ subject to constraints } x^T A \geq p 1_{1,n}, x \geq 0, x^T 1_{m,1} = 1$, for player 1 and $\min p \text{ subject to constraints } A y \leq p 1_{m,1}, y \geq 0, 1_{1,n} y = 1$, for player 2. These linear programming problems are *dual* to each other.

- **Practical tip**: We can also insure that $p \geq 0$ by simply adding a constant to every payoff so that the table is nonnegative, computing the strategies and then subtracting the constant from the computed optimal payoff.

- Let’s set up all three examples as LP problems, both explicitly and in matrix form, and solve them with Matlab to determine optimal strategies for each game. Check requirements of *linprog* first.
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The Original Model Problem:

Two companies compete for the bulk of a shared market for a certain product and plan to execute one of three strategies. Both marketing departments analyzed them and have arrived at essentially the same figures for outcomes.

- The three strategies are:
  - Better packaging.
  - An advertising campaign.
  - Slight price reduction.

- Suppose there is considerable uncertainty about the payoff in the case that both players make a slight reduction in price. How could we clearly illustrate the effect of changes in the payoff on the weight that one of the companies puts on this strategy?
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