JDEP 384H: Numerical Methods in Business

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Department of Mathematics

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110 Kaufmann Center

Outline

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Definition of An Asian Option:

Here is one type of Asian (call) option, where a daily average of the stock is computed over the life of the option. Instead of using price $S(T)$ of the option, one uses the average value of the option in the payoff formula.

- Result: the payoff curve gives
  $f_T = \max\left\{0, \frac{1}{N}\sum_{i=1}^{N} S(t_i) - K\right\}$ where K is the strike price and T consists of N days.
- Thus the actual path of a particular daily history of the stock price influences the payoff and hence the price of the option. To estimate the price f discount the average of the computed values f_T over various stock paths.
- Control variates can be helpful. Use $C = \sum_{i=1}^{N} S(t_i)$ as control. Compute expected value by discounting each term to expiry T.

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Example Calculations:

Use simple Monte Carlo and control variates to estimate the value of an Asian call with same data as in previous examples. Use norm_fit to obtain confidence intervals.

```
> AsianExampleData
> Spaths = AssetPath(S0,mu,sigma,T,nsteps,nreps);
> SFinal = mean(Spaths(:,2:nsteps+1)')';
> prices = exp(-r*T)*max(0,SFinal - K);
> [price, V, CI] = norm_fit(prices)
> bseurcall(S0,K,r,T,0,sigma,0) % just for comparison
> Cn = nsteps*SFinal; % control variate
> muC = mean(Cn); % expected value of C
> S = cov([prices,Cn]); % calculate covariance matrix
> bta = -S(2,1)/S(2,2) % estimate bta
> pricesC = prices + bta*(Cn - muC);
> [prices, V, CI] = norm_fit(pricesC)
> [smplmu,smplstdv,muci] = norm_fit(pricesC,alpha)
```

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Barrier Options

What They Are:

Option in which there is an agreed barrier S_b, which may or may not be reached by the price S of the stock in question.

- Knock-out option: contract is voided if the barrier value is reached at any time during the whole life of option.
- Knock-in option: option is activated only if the barrier is reached.
- $S > S_b$: a down option.
- $S < S_b$: an up option.
- Analytic formulas are known for certain barrier options, including a down-and-out put.

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How It's Done:

- Given a random walk of $\mathbf{S}$ of length N with $S(t_N) = T$, the payoff curve gives $f_T = \max\{0, (S(t_N) - K) I(\mathbf{S})\}$ where K is the strike price and the indicator function $I(\mathbf{S})$ is zero if any $S(t_j) < S_b$ and one otherwise.
- Of course, f_T is a r.v. But the drift for this asset should be the risk-free rate r. So all we have to do is average the payoffs over various stock price paths to time T, then discount the average to obtain an approximation for f_0.

Pricing a Barrier Option by Monte Carlo Methods

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Example Calculations

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Use simple Monte Carlo to estimate the value of a down-and-out put with same data as in previous examples and $S_b = 80$. Use norm_fit to obtain confidence intervals.

```
> Sb = 80
> truemean = DownOutPut(S0,K,r,T,0,sigma,Sb)
> bseurput(S0,K,r,T,0,sigma,0) % just for comparison
> randn('state',0)
> Spaths = AssetPath(S0,mu,sigma,T,nsteps,nreps);
> SFinal = Spaths(:,nsteps+1);
> Indicator = (sum((Spaths<Sb)')')==0;
> prices = exp(-r*T)*max(0,(K-SFinal).*Indicator);
> [smplmu,smplvar,muci,varci] = norm_fit(prices,alpha)
> % now lower the barrier and see what happens
```

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A Model Problem

Business Competition as a Game:

Two companies compete for the bulk of a shared market for a certain product and plan to execute one of three strategies. Both marketing departments analyzed them and have arrived at essentially the same figures for outcomes.

- The three strategies are:
  - Better packaging.
  - An advertising campaign.
  - Slight price reduction.
- We assume:
  - Payoffs for this competition (game) are known to all.
  - Players are rational players who play to win.
  - Neither side has advance knowledge of the opponent's play.
  - Whatever one side gains, the other loses (*zero-sum game*.)

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The Payoff Table:

Here is the payoff table for player (company) 1 in percentage gain or loss of share:

Strategy		Player 2: 1	2	3
Player 1	1	2	3	1
	2	1	4	0
	3	3	-2	-1
- There is no need to give player (company) 2's payoff table since it is the negative of player 1's table by the zero-sum property.
- Now what?

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Stragegies

The Simplest Approach:

If any one stragegy is dominated by another, i.e., worse no matter what the opponent does, then discard it.

- Apply this stragegy to the model problem and bring it to a resolution. What is the value of the game, i.e., the payoff if each player executes its optimal strategy?
- The entry of this payoff is a *saddle point* in the revised payoff table, i.e., a minimum in its row and maximum in its column.
- Is this game fair (i.e., optimum payoff is zero)?
- BUT....What if the payoff table looks like this:

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- Why won't eliminating dominated strategies resolve the game?

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The Next Simplest Approach (Minimax/Maximin):

The idea is to look at the worst outcomes for each move the opponent might make, then choose the most favorable of worst payoffs. For player 1, this means maximizing the minimum payoff, for player 2, this means minimizing the maximum payoff.

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Use *mixed strategies* instead of pure strategies, i.e., a probability vector (x_1, x_2, x_3) for player 1 ($y = (y_1, y_2, y_3)$ for player 2) that maximizes (minimizes) the payoff for all possible plays by the opponent.

- If the payoff table is converted into a matrix A ($m \times n$ in general), then the payoff for any pair of mixed strategies is

$$p = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = \mathbf{x}^T A \mathbf{y}$$

- Player 1's goal: Find probability vector $\mathbf{x}$ solving $\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}^T A \mathbf{y}$.
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