Chapter 4: Numerical Integration: Deterministic and Monte Carlo Methods

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Department of Mathematics

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110 Kaufmann Center
1. Chapter 4: Numerical Integration: Deterministic and Monte Carlo Methods
   - BT 4.1: Numerical Integration
   - BT 4.2: Monte Carlo Integration
   - BT 4.3: Generating Pseudorandom Variates
   - BT 4.4: Setting the Number of Replications
   - BT 4.5: Variance Reduction Techniques
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Outline

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Problem:

We estimate a mean by a sample mean \( \bar{X} (n) = \frac{1}{n} \sum_{i=1}^{n} X_i \) approximating true mean \( \mu \) and variance by sample variance \( S^2 (n) = \frac{1}{n-1} \sum_{i=1}^{n} [X_i - \bar{X} (n)]^2 \) approximating true variance \( \sigma^2 \).

- We know that \( |\bar{X} (n) - \mu| \leq z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \approx z_{1-\alpha/2} \frac{S (n)}{\sqrt{n}} \) at the \((1 - \alpha)\) confidence level. (See Lecture 6.)

- So, to bound the error by \( \beta \) with the same confidence, require that \( z_{1-\alpha/2} \frac{S(n)}{\sqrt{n}} \leq \beta \).

- A little calculation shows that to bound the relative error by \( \beta \), require that \( \left( z_{1-\alpha/2} \frac{S(n)}{\sqrt{n}} \right) / |\bar{X} (n)| \leq \beta / (1 + \beta) \)

- These may require large \( n \), which could be a problem. (See what \( \beta = 0.1 \) entails.) Possible solution: reduce variance of sample.
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Setting Accuracy

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Variance Reduction 1: Antithetic Variates

Basic Idea:
To estimate $E[X] = \mu$:

- Generate paired r.v.’s $(X_1^{(i)}, X_2^{(i)})$, $i = 1, \ldots, n$ with horizontal independence, but not necessarily vertical independence.
- Construct pairs so that $X_1^{(i)}, X_2^{(i)}$ are negatively correlated.
- Use pair-averaged samples $X^{(i)} = \left( X_1^{(i)} + X_2^{(i)} \right) / 2$. Reason:

$$\text{Var} \left( \frac{\bar{X}(n)}{2} \right) = \frac{\text{Var} \left( X_1^{(i)} \right) + \text{Var} \left( X_2^{(i)} \right) + 2 \text{Cov} \left( X_1^{(i)}, X_2^{(i)} \right)}{4n}$$

- Hope this reduces the variance of the sample.
- Practical pointer: If $X = g(U)$ are supposed to be generated by uniform $U(0,1)$ samples $U_i$, try $X_1^{(i)} = g(U_i)$ and $X_2^{(i)} = g(1 - U_i)$. IF $g(u)$ is monotone increasing, this works! Caution: without some restrictions, it can make things worse!
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$$\text{Var} \left( \bar{X} \right) = \frac{\text{Var} \left( X_1^{(i)} \right) + \text{Var} \left( X_2^{(i)} \right) + 2 \text{Cov} \left( X_1^{(i)}, X_2^{(i)} \right)}{4n}$$

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\text{Var} \left( \bar{X} (n) \right) = \frac{\text{Var} \left( X_1^{(i)} \right) + \text{Var} \left( X_2^{(i)} \right) + 2 \text{Cov} \left( X_1^{(i)}, X_2^{(i)} \right)}{4n}
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Returning to our Monte Carlo integration example, find the $n$ that will give absolute error at most 0.01 at the 95% confidence level using antithetic variates. Experiment with this Matlab code.

```matlab
> mu = exp(1)-1
> rand('seed',0)
> alpha = 0.05 % 95 percent confidence level
> zalpha = stdn_inv(1 - alpha/2)
> n = 100
> U1 = rand(n,1);
> U2 = 1-U1;
> Xn = 0.5*(exp(U1)+exp(U2));
> Xbar = mean(Xn)
> sigma2 = var(Xn)
> bta = zalpha*sqrt(sigma2/n)
```
Basic Idea:

To estimate $E[X] = \mu$:

- $\leftarrow$ Find a random variable $C$, with known mean $\mu_C$ and form r.v. $X_C = X + \beta (C - \mu)$.
- Have $E[X_C] = E[X] = \mu$.
- Have $\text{Var}([X_C]) = \text{Var}(X) + \beta^2 \text{Var}(C) + 2\beta \text{Cov}(X, C)$.
- So if $2\beta \text{Cov}(X, C) + \beta^2 \text{Var}(C) < 0$, we get reduction with optimum at $\beta = \beta^* = -\frac{\text{Cov}(Y, C)}{\text{Var}(C)}$ (why?), with variance $(1 - \rho^2 (X, C)) \text{Var}(X)$. In practice, we estimate $\beta^*$ experimentally.
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> alpha = 0.05 % 95 percent confidence level
> zalpha = stdn_inv(1 - alpha/2)
> n = 100
> Un = rand(n,1);
> Cn = 1+(exp(1)-1)*Un; % Control variate based on
linear approxn
> muC = 1+(exp(1)-1)*0.5 % Expected value of C
> bttta = -0.5; % postive correlation, so negative beta
> Xn = exp(Un);
> XCbar = mean(Xn+bttta*(Cn-muC))
> sigma2 = var(Xn+bttta*(Cn-muC))
> bta = zalpha*sqrt(sigma2/n)
```