JDEP 384H: Numerical Methods in Business

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Department of Mathematics

Lecture 22, April 5, 2007
110 Kaufmann Center

Outline

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1. Chapter 4: Numerical Integration: Deterministic and Monte Carlo Methods
   - BT 4.1: Numerical Integration
   - BT 4.2: Monte Carlo Integration
   - BT 4.3: Generating Pseudorandom Variates
   - BT 4.4: Setting the Number of Replications
   - BT 4.5: Variance Reduction Techniques

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Setting Accuracy

Problem:

We estimate a mean by a sample mean $\bar{X}(n) = \frac{1}{n}\sum_{i=1}^{n} X_i$ approximating true mean μ and variance by sample variance $S^2(n) = \frac{1}{n-1}\sum_{i=1}^{n}\left[X_i - \bar{X}(n)\right]^2$approximating true variance σ^2.

- We know that $\left|\bar{X}(n) - \mu\right| \leq z_{1-\alpha/2}\dfrac{\sigma}{\sqrt{n}} \approx z_{1-\alpha/2}\dfrac{S(n)}{\sqrt{n}}$ at the $(1-\alpha)$ confidence level. (See Lecture 6.)
- So, to bound the error by β with the same confidence, require that $z_{1-\alpha/2}\frac{S(n)}{\sqrt{n}} \leq \beta$.
- A little calculation shows that to bound the relative error by β, require that $\left(z_{1-\alpha/2}\frac{S(n)}{\sqrt{n}}\right) / \left|\bar{X}(n)\right| \leq \beta/(1+\beta)$
- These may require large n, which could be a problem. (See what $\beta = 0.1$ entails.) Possible solution: reduce variance of sample.

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Variance Reduction 1: Antithetic Variates

Basic Idea:

To estimate $E[X] = \mu$:

- Generate paired r.v.'s $\left(X_1^{(i)}, X_2^{(i)}\right)$, $i = 1, \ldots, n$ with horizontal independence, but not necessarily vertical independence.
- Construct pairs so that $X_1^{(i)}, X_2^{(i)}$ are negatively correlated.
- Use pair-averaged samples $X^{(i)} = \left(X_1^{(i)} + X_2^{(i)}\right)/2$. Reason:

$$\operatorname{Var}\left(\bar{X}(n)\right) = \frac{\operatorname{Var}\left(X_1^{(i)}\right) + \operatorname{Var}\left(X_2^{(i)}\right) + 2\operatorname{Cov}\left(X_1^{(i)}, X_2^{(i)}\right)}{4n}$$

- Hope this reduces the variance of the sample.
- Practical pointer: If $X = g(U)$ are supposed to be generated by uniform $U(0,1)$ samples U_i, try $X_1^{(i)} = g(U_i)$ and $X_2^{(i)} = g(1 - U_i)$. IF $g(u)$ is monotone increasing, this works! Caution: without some restrictions, it can make things worse.

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Calculations

Returning to our Monte Carlo integration example, find the *n* that will give absolute error at most 0.01 at the 95% confidence level using antithetic variates. Experiment with this Matlab code.

```
> mu = exp(1)-1
> rand('seed',0)
> alpha = 0.05 % 95 percent confidence level
> zalpha = stdn_inv(1 - alpha/2)
> n = 100
> U1 = rand(n,1);
> U2 = 1-U1;
> Xn = 0.5*(exp(U1)+exp(U2));
> Xbar = mean(Xn)
> sigma2 = var(Xn)
> bta = zalpha*sqrt(sigma2/n)
```

Variance Reduction 2: Control Variates

Basic Idea:

To estimate $E[X] = \mu$:

- <+-> Find a random variable C, with known mean μ_C and form r.v. $X_C = X + \beta (C - \mu)$.
- Have $E[X_C] = E[X] = \mu$.
- Have $\text{Var}([X_C]) = \text{Var}(X) + \beta^2 \text{Var}(C) + 2\beta \text{Cov}(X, C)$.
- So if $2\beta \text{Cov}(X, C) + \beta^2 \text{Var}(C) < 0$, we get reduction with optimum at $\beta = \beta^* = -\text{Cov}(Y, C) / \text{Var}(C)$ (why?) , with variance $(1 - \rho^2 (X, C)) \text{Var}(X)$. In practice, we estimate β^* experimentally.

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> n = 100
> Un = rand(n,1);
> Cn = 1+(exp(1)-1)*Un; % Control variate based on
linear approxn
> muC = 1+(exp(1)-1)*0.5 % Expected value of C
> btta = -0.5; % postive correlation, so negative beta
> Xn = exp(Un);
> XCbar = mean(Xn+btta*(Cn-muC))
> sigma2 = var(Xn+btta*(Cn-muC))
> bta = zalpha*sqrt(sigma2/n)
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