JDEP 384H: Numerical Methods in Business

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Lecture 2, January 11, 2007
110 Kaufmann Center

Outline

The Arithmetic

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices. We define

- If A and B are of the same size, we can add them:

$$A + B = [a_{ij} + b_{ij}]$$

- If c is a scalar (i.e., number) we can *scalar multiply* a matrix A by it:

$$cA = [ca_{ij}]$$

- If A is $m \times p$ and B is $p \times n$, we can *matrix multiply* them to get an $m \times n$ matrix AB given by

$$AB = \left[\sum_{k=1}^{p} a_{ik} b_{kj} \right]$$

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Transposes

Transpose of $m \times n$ matrix $A = [a_{ij}]$ is $n \times m$ matrix

$$A^T = [a_{ji}],$$

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$

Laws of transposes

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ (AB)^T &= B^T A^T \\ (cA)^T &= cA^T \\ \left(A^T\right)^T &= A \end{aligned}$$

Special Matrices

- Zero matrix $\mathbf{0}$ (or $\mathbf{0}_{m,n}$) is an $m \times n$ matrix whose every entry is the scalar 0, e.g., $\mathbf{0}_{3,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- Identity matrix I (or I_n) is a square $n \times n$ matrix whose (i, i)th entries are 1 and all others 0, e.g., $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Diagonal matrix D is square $n \times n$ matrix whose (i, i)th entries are nonzero and all others are zero, e.g., $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- (Upper) triangular matrix U is square $n \times n$ whose (i, j)th entries are zero if $i > j$ (further down than over), e.g.,
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Matrix Powers

Given square $n \times n$ matrix A

Inverse of A

is (unique if it exists at all!) $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I = A^{-1}A.$$

Note: Laws of inverses have same form as laws of transposes.

Positive exponent r:

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Laws of Exponents

$$A^{r+s} = A^r A^s$$
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Outline

1. Linear Algebra
   - Dot Products and Norms
   - Eigenvalue Problems

Definitions

We know how to measure the size of a scalar quantity x: use $|x|$. Thus, we have a way of thinking about things like the *size* of an error.

Question: How do we measure the size of vectors or matrices?

Answer: We use some kind of yardstick called a **norm** that assigns to each vector $\mathbf{x}$ a non-negative number $\|\mathbf{x}\|$ subject to the following norm laws for arbitrary vectors $\mathbf{x}, \mathbf{y}$ and scalar c:

- For $\mathbf{x} \neq \mathbf{0}$, $\|\mathbf{x}\| > 0$ and for $\mathbf{x} = \mathbf{0}$, $\|\mathbf{x}\| = 0$.
- $\|c\mathbf{x}\| = |c| \|\mathbf{x}\|$.
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Examples are to be found in our Linear Algebra Notes file: LinearAlgebraLecture-384H.pdf. Of course, Matlab knows all about the various norms we use on vectors or matrices.

Outline

1. Linear Algebra
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Eigenthings

Another big idea that we'll make some use of later in the course:

Eigenvalue for square matrix A

is a scalar λ such that for some NONZERO vector $\mathbf{x}$, called an eigenvector corresponding to λ,

$$A\mathbf{x} = \lambda\mathbf{x}.$$

And so??? Again, this is covered more thoroughly in our Linear Algebra Notes file: LinearAlgebraLecture-384H.pdf. The basic idea: eigenvalues are a set of numbers that tell us about the way multiplication by A stretches or shrinks vectors. Again, Matlab knows all about finding eigenvalues and eigenvectors.

An Example that Puts it All Together

Example

Suppose two toothpaste companies compete for customers in a fixed market in which each customer uses either Brand A or Brand B. Suppose also that a market analysis shows that the buying habits of the customers fit the following pattern in the quarters that were analyzed: each quarter (three-month period), 30% of A users will switch to B, while the rest stay with A. Moreover, 40% of B users will switch to A in a given quarter, while the remaining B users will stay with B. If we *assume* that this pattern does not vary from quarter to quarter, we have an example of what is called a Markov *chain model.* Express the data of this model in matrix–vector language.

Let's discuss this at the board and explore some ideas...

Example (continued)

Now we see that the state vectors and transition matrices

$$\mathbf{x}^{(k)} = \begin{bmatrix} a_k \\ b_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

play an important role. And indeed they do, for in light of our interpretation of a linear system as a matrix product, we see that the two equations can be written simply as $\mathbf{x}^{(1)} = A\mathbf{x}^{(0)}$, or more generally as $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$. Any system of successive state vectors $\mathbf{x}^{(0)}, \mathbf{x}^{(1)} \ldots$ generated by such a formula is called a (linear) **discrete dynamical system**. A little more calculation shows that

$$\mathbf{x}^{(2)} = A\mathbf{x}^{(1)} = A \cdot (A\mathbf{x}^{(0)}) = A^2\mathbf{x}^{(0)}$$

and in general,

$$\mathbf{x}^{(k)} = A\mathbf{x}^{(k-1)} = A^2\mathbf{x}^{(k-2)} = \cdots = A^k\mathbf{x}^{(0)}.$$

Example (continued)

O.K., let's go to Matlab:

```
A = [0.7 0.4; 0.3, 0.6]
x = [1;0]
% cursor up the next line repeatedly for a pattern...
x = A*x
% now save your latest x
x0 = x
x = [0;1]
% now do repetitions again and when done...
x - x0
% finally, try this:
[P, D] = eig(A)
v = P(:,1)
A*v
v/sum(v)
```