Midterm Exam JDEP 384H: Numerical Methods for Business

Name: ________________________ **Score:** __________

Instructions: Write out your solutions on the paper provided. Show your work and give reasons for your answers. NO notes, calculators, laptops, cell phones or other electronic equipment allowed. Each problem is worth 20 points *for a total of* 100 *points.* Work problem 1 and any four of the remaining five problems. *Clearly indicate which four you want graded.* Give exact answers (like $1/3$ or $\sqrt{2}$) where calculations are expected.

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1. Answer True or False for each of the following statements. If false, correct the statement (nontrivially).

(a) An eigenvector for the (square) matrix A is a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ called an eigenvalue of A.

(b) If D_M is the modified duration of a bond and C is its convexity at rate λ, then $\delta P = P(\lambda + \delta\lambda) - P(\lambda)$ has second order approximation $-D_M P\,\delta\lambda + \frac{PC}{2}(\delta\lambda)^2$.

(c) The correlation between two random variables X and Y with covariance C and standard deviations a, b, respectively, is $C/(ab)$

(d) The VaR of a portfolio whose returns are normally distributed is completely determined by its value, volatility and a time horizon.

(e) For a generic option of price $f(S,t)$ the "Greek" $\Delta = \partial f(S,t)/\partial S$ measures the sensitivity of the option price to small changes in time.

(f) The price of an American call with dividends may fall below the payoff curve.

(g) In order to construct a binomial lattice for the price of an option on a stock, we need to know the actual probability that the stock will increase in value in the lattice time period.

Solution. (a) False. Corrected: An eigenvector for the (square) matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ called an eigenvalue of A.

(b) True.

(c) True.

(d) False. Corrected: The VaR of a portfolio whose returns are normally distributed is completely determined by its value, volatility, a time horizon and a confidence level.

(e) False. Corrected: For a generic option of price $f(S,t)$ the "Greek" $\Delta = \partial f(S,t)/\partial S$ measures the sensitivity of the option price to small changes in price.

(f) False. Corrected: The price of an European call with dividends may fall below the payoff curve.

(g) False. Corrected: In order to construct a binomial lattice for the price of an option on a stock, we need to know the risk-neutral probability that the stock will increase in value in the lattice time period.

2. Consider the following linear programming problem in canonical form:

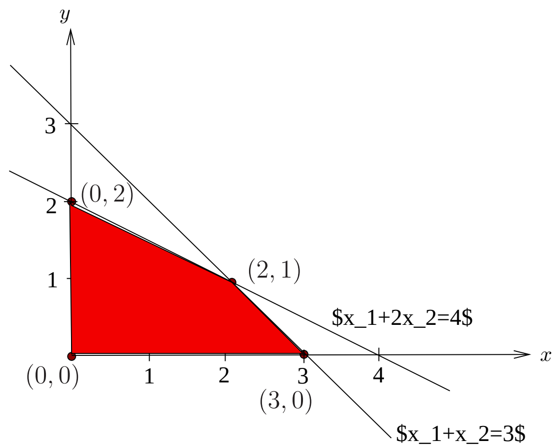
$$\text{Maximize } Z = 2x_1 + 3x_2$$

subject to the constraints $x_i \geq 0$, $i = 1,2$, and

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1 + x_2 &\leq 3. \end{aligned}$$

(a) Solve this problem directly by the graphical method.
(b) Express it in standard form (only inequalities are $x_i \geq 0$.)

Solution. (a) Solve the simultaneous equations that define the boundaries of the inequalities: subtracting gives $x_2 = 4 - 3 = 1$, so that $x_1 = 3 - 1 = 2$. Now draw a picture of the feasible set:



We see that the only feasible corner points are $(0,0)$, $(3,0)$, $(2,1)$ and $(0,2)$. Evaluate the objective function at each to obtain

$$\begin{aligned} Z(0,0) &= 0 \\ Z(3,0) &= 6 \\ Z(2,1) &= 7 \\ Z(0,2) &= 6. \end{aligned}$$

So the maximum value is $Z = 7$ and it occurs at the point $x_1 = 2$, $x_1 = 1$.

(b) Standard form: Minimize $-2x_1 - 3x_2$ (or Maximize $2x_1 + 3x_2$) subject to constraints $x_1 + 2x_2 + x_3 = 4$, $x_1 + x_2 + x_4 = 3$, and $x_j \geq 0$, $j = 1,2,3,4$.

3. Consider the system of equations

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ x_1 + x_2 + x_3 &= -2. \end{aligned}$$

(a) Express this system in matrix form $A\mathbf{x} = \mathbf{b}$.
(b) Write out explicitly (no matrices or vectors), the system defined by $A^T A\mathbf{x} = A^T\mathbf{b}$.

Solution.
(a) We define the matrix A and vectors $\mathbf{x}$ and $\mathbf{b}$ by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix},$$

so that the system is $A\mathbf{x} = \mathbf{b}$ or $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(b) We calculate

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

and

$$A^T \mathbf{b} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$$

so that the normal equations in terms of the unknowns (x_1, x_2, x_3) are

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= -1 \\ x_1 + 2x_2 &= 4 \\ 2x_1 + 3x_3 &= 1. \end{aligned}$$

4. Consider a weighted portfolio of two bonds with face value 100: B_1 is a 1 year zero coupon bond with rate of return 6% and B_2 is a 3 year bond with annual coupons rate of 8%.

(a) Given the current risk-free interest rate is r, write formulas for the present value of B_1 and B_2.

(b) If you wanted to build a portfolio whose weighted duration is two years, what would be the system of equations you would have to solve (don't do it.) You may assume the duration of B_2 is about 2.78.

Solution.

(a) The formula for the bond B_1 is, assuming r is given in decimal form (otherwise use $r/100$),

$$PV(B_1) = \frac{100}{1+r}.$$

For the bond B_2 the formula is

$$PV(B_2) = \frac{8}{1+r} + \frac{8}{(1+r)^2} + \frac{108}{(1+r)^3}.$$

(b) We will assume that negative weightings are allowed, since we are not told to the contrary. Therefore, if the weightings are given by w_1 for B_1 and w_2 for B_2, then we must have the sum be 1 and weighted the duration of the portfolio matched. Now the duration of a zero coupon bond is simply its time to maturity. Hence the system to be solved is

$$\begin{aligned} w_1 + w_2 &= 1 \\ w_1 + 2.78 w_2 &= 2. \end{aligned}$$

5. A portfolio consists of two assets with weights w_1 and w_2 and no short positions. Variance of the two assets are $\sigma_1 = 0.3$ and $\sigma_2 = 0.6$, and the covariance is $\sigma_{12} = -0.1$. Expected returns are $r_1 = 0.04$ and $r_2 = 0.08$.

(a) Find the expected return and variance of the portfolio in terms of w_1 and w_2.

(b) Write out the system you would have to solve to find the portfolio with minimum volatility (do not solve it.)

Solution.

(a) Let the weight vector be $\mathbf{w} = (w_1, w_2)$, the expected return vector $\mathbf{r} = (0.04, 0.08)$ and covariance matrix $S = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.6 \end{bmatrix}$. The expected return of the weighted portfolio is given by

$$\mathbf{r}^T\mathbf{w} = [0.04, 0.08] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.04w_1 + 0.08w_2.$$

The variance is calculated as

$$\begin{aligned} \mathbf{w}^T S\mathbf{w} &= [w_1, w_2] \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= [w_1, w_2] \begin{bmatrix} 0.3w_1 - 0.1w_2 \\ -0.1w_1 + 0.6w_2 \end{bmatrix} \\ &= w_1 (0.3w_1 - 0.1w_2) + w_2 (-0.1w_1 + 0.6w_2) \\ &= 0.3w_1^2 - 0.2w_1w_2 + 0.6w_2^2. \end{aligned}$$

(b) In order to find the portfolio of minimum variance, given that no short positions are allowed, we would have to solve for (w_1, w_2) in the quadratic programming program

$$\begin{aligned} &\text{Min } \mathbf{w}^T S\mathbf{w} = 0.3w_1^2 - 0.2w_1w_2 + 0.6w_2^2 \\ &\text{Sbj.} \\ &w_1 + w_2 = 1 \\ &w_1 \geq 0, w_2 \geq 0. \end{aligned}$$

6. Consider the stochastic differential equation for a risky asset $S(t)$:

$$dS = \mu S\,dt + \sigma S\,dW.$$

(a) Explain what each term is.

(b) Calculate the mean and variance of dS.

Solution.

(a) The term $S = S(t)$ is a random variable that represents the price of a risky asset. The terms dS and dW are stochastistic differentials; dS is defined by the formula and dW is understood to be a time dependent stochastic random variable (the differential of the Weiner process) that is normally distributed with mean 0 and standard deviation $\sqrt{dt}$. Also, dt is a deterministic variable, namely the differential of time, while μ is a positive constant known as a drift term. Finally, σ is also a postive constant known as the volatility of the asset price.

(b) To calculate mean, we assume that $S(t)$, μ, σ and dt are known (hence non-random) terms and use the standard rules to calculate

$$\begin{aligned} E[dS] &= E[\mu S\,dt + \sigma S\,dW] \\ &= E[\mu S\,dt] + E[\sigma S\,dW] \\ &= \mu S\,dt + \sigma S\,E[dW] \\ &= \mu S\,dt. \end{aligned}$$

The variance is calculated to be

$$\begin{aligned} \text{Var}[dS] &= \text{Var}[\mu S\,dt + \sigma S\,dW] \\ &= \sigma^2 S^2\,\text{Var}[\sigma S\,dW] \\ &= \sigma^2 S^2 dt. \end{aligned}$$

Take-Home Portion

Points: 35

Due Date: Friday, March 23, 5:00 p.m. for hard copy, 11:00 p.m. for email.

Instructions: Show your work and give reasons for your answers. There is to be absolutely no consultation of any kind with anyone else other than me about the exam. If there are points of clarification or corrections, I will post them on our message board. ALL materials used in your work that have not been provided by me for this course must be explicitly credited in your write-up. Point values are indicated. You may send an email document (preferably a pdf file, but Word documents will be accepted — in all cases it must be a *single* document) or hand in hard copy at my office.

(15 pts) **1.** You are to build a portfolio from four available coupon bonds, all with face value of $\$1,000$, coupon rates of 5, 6, 7, and 8 percent and period 2, maturities 2, 4, 6, 7 years, respectively, as a weighted combination of these bonds. No short positions are allowed. The prevailing risk-free interest rate is 7%. We wish to find a portfolio whose weighted convexity is maximized subject to the constraints that the weighted duration is at most the average of the shortest and longest bond duration, and no more than 40% of the investment is in bonds of maturities greater than 5 years.

(a) Express this problem as a linear programming problem.

(b) Use Matlab to solve it.

Solution.

(a) Let the bonds be labeled as B_1, B_2, B_3, and B_4 with weighting vector $\mathbf{w} = (w_1, w_2, w_3, w_4)$ for this portfolio, so that $\mathbf{w} \geq \mathbf{0}$ since short positions are not allowed. Here w_i is the weight of B_i in the portfolio. Let C_i be the convexity and D_i the duration of the ith bond B_i.

The objective function is

$$Z = C_1w_1 + C_2w_2 + C_3w_3 + C_4w_4.$$

We can maximize this by minimizing $-Z$, which we do for use in Matlab.

The first constraint is that the weighted duration not exceed the average of the shortest and longest duration, that is,

$$D_1w_1 + D_2w_2 + D_3w_3 + D_4w_4 \leq \frac{1}{2}(D_1 + D_4).$$

Here we're assuming bonds of longer maturity will have larger duration. We have to confirm this computationally.

The second constraint is no more than 40% of the portfolio be vested in bonds of maturities greater than 5 years. Thus

$$w_3 + w_4 \leq 0.4.$$

The third constraint is that the weights be weights, i.e., sum to one

$$w_1 + w_2 + w_3 + w_4 = 1.$$

Hence the total problem is

$$\begin{aligned}
&\text{Min } -Z = -C_1w_1 - C_2w_2 - C_3w_3 - C_4w_4 \\
&\text{Sbj.} \\
&D_1w_1 + D_2w_2 + D_3w_3 + D_4w_4 \leq \frac{1}{2}(D_1 + D_4) \\
&w_3 + w_4 \leq 0.4 \\
&w_1 + w_2 + w_3 + w_4 = 1 \\
&w_1 \geq 0, w_2 \geq 0, w_3 \geq 0, w_4 \geq 0.
\end{aligned}$$

(b) The Matlab script that solves it is listed below, followed by the output of the script.

```
% Script:  TakeHome1b.m
% description:  this file calculates the
% the numbers required by Take Home Problem 1(b).
r = 0.07;
r2 = r/2;
face = 1000;
cpns2 = face*[0.05, 0.06, 0.07, 0.08]/2;
matur = [2,4,6,7];
% set up the cash flows for each bond, excluding 0th
B1 = [cpns2(1)*ones(1,2*(matur(1))-1),face+cpns2(1)];
B2 = [cpns2(2)*ones(1,2*(matur(2))-1),face+cpns2(2)];
B3 = [cpns2(3)*ones(1,2*(matur(3))-1),face+cpns2(3)];
B4 = [cpns2(4)*ones(1,2*(matur(4))-1),face+cpns2(4)];
% find durations
dur1 = cfdur(B1,r2);
dur2 = cfdur(B2,r2);
dur3 = cfdur(B3,r2);
dur4 = cfdur(B4,r2);
% find convexities
conv1 = cfconv(B1,r2);
conv2 = cfconv(B2,r2);
conv3 = cfconv(B3,r2);
conv4 = cfconv(B4,r2);
% coefficient vector for objective function
c = -[conv1,conv2,conv3,conv4]'
% right-hand side vector
b = [0.5*(dur1 + dur4); 0.4]
% coefficient matrix
A = [dur1, dur2, dur3, dur4; 0 0 1 1]
% make allowance for the equality constraint
[wts,minusoptconv] = linprog(c,A,b,[],[],[1 1 1 1],1)
```

The output:

```
octave:11> TakeHome1b
c =
-17.757
-58.483
-113.742
-142.373
b =
7.46496
0.40000
A =
3.85291 7.21402 10.00155 11.07702
0.00000 0.00000 1.00000 1.00000
wts =
0.38507
0.21493
```

```
0.00000
0.40000
minusoptconv = -76.357
octave:12> quit
```

From this we see that the correct weighting is $\mathbf{w} = (0.385, 0.215, 0, 0.4)$, which yields a maximum convexity of 76.357.

(20 pts) **2.** You are given that the risk-free interest rate is 10%, and that a put option on a stock with volatility $\sigma = 0.4$ has 6 months from now ($t = 0$) to expiry, at which point the strike price of the option is $50.

(a) Use Matlab tools to make a plot of the values of European and American put options at $t = 0$ as functions of stock price S, and put the payoff curve for this option on the same graph. Label the curves.

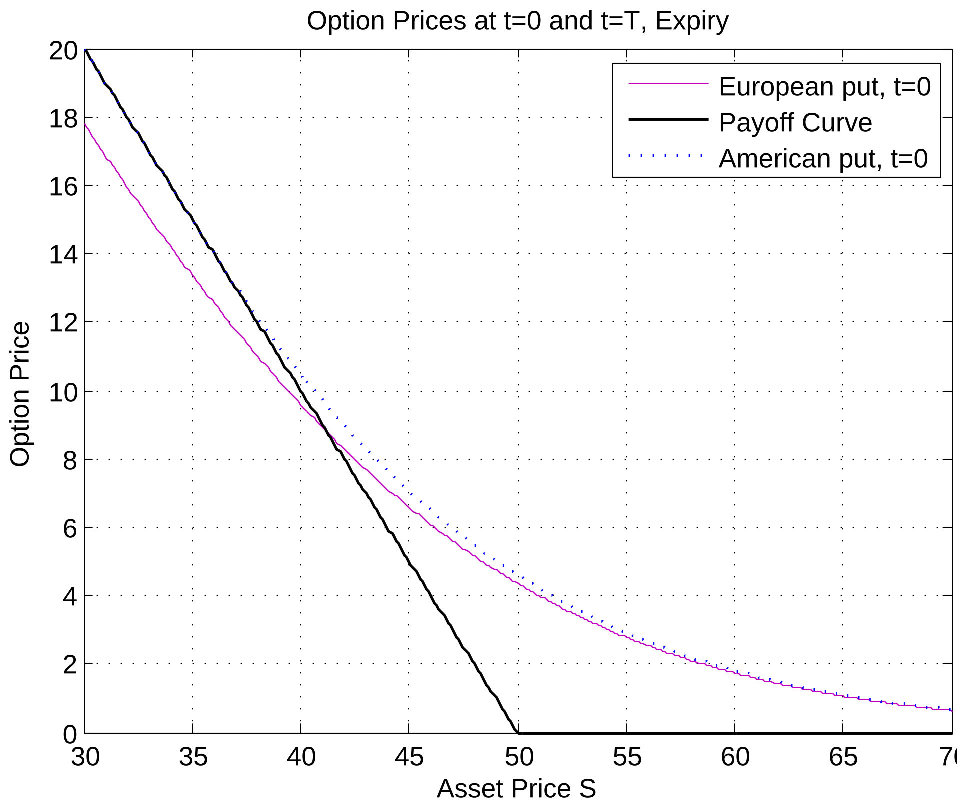
(b) Use the graphs or any other Matlab tools to estimate the value of S at which one of the curves crosses the payoff curve and the value of S where the other just touches the payoff curve.

Solution.

(a) The script that generates the data and graphs is listed below, followed by the graph. We first generated very crude graphs to get an idea of the necessary range. The range $30 \leq S \leq 70$ appears to capture all the features we need. Graphing the European option is fast, but graphing the American option to a significant degree of accuracy is slow. The text examples used lattices of size $N = 1000$ or twice that for 3-4 digits of accuracy. So we'll try that, knowing it will take a little time.

```
% script:  TakeHome2.m
% This script works parts (a) and (b) of
% the take home exam
mystartup % make right directories available
t = 0 % current time
K = 50 % strike price
r = 0.1 % current risk-free interest rate
T = 6/12 % expiry
sigma = 0.4 % volatility
S = 30:.1:70; % interval of interest
D0 = 0 % no dividends
EP = bseurput(S,K,r,T,t,sigma,D0);
plot(S,EP) % plot European put
hold on % so we can overlay graphs
grid % so we can discriminate
payoff = max(K-S,0); % payoff curve
plot(S,payoff) % plot payoff curve
% now build the American put, term by term
% N is size of lattice...  from examples in
% text, N in range of 1000-2000 should be
% good for three digits (see text, p.  414-417)
% start with something less in interests of time
N = 1000
AP = S;
for ii = 1:length(S)
AP(ii) = LatticeAmPut(S(ii),K,r,T,sigma,N,D0);
end
plot(S,AP)
```

The plot looks like this:



(b) There are numerous ways we can proceed. It is clear from the graph that the European curve definitely crosses the payoff curve, and we can see that it is near 41. We know the graph is accurate, so we can nail down the crossing point by something like this:

```
ndx = find(EP >= payoff);
S(ndx(1))
ans = 41.300
```

So our answer is $S = 41.3$ is the crossing point.

The American put is tougher, because we not sure what lattice size to use for high accuracy. We could do something like this:

```
ndx = find(AP >= payoff + 0.01); % this was with N=100
S(ndx(1))
ans = 36.2
% clinch it with a more refined calculation
payoff(ndx(1))
ans = 13.800
LatticeAmPut(S(ndx(1)),K,r,T,sigma,1000)
```

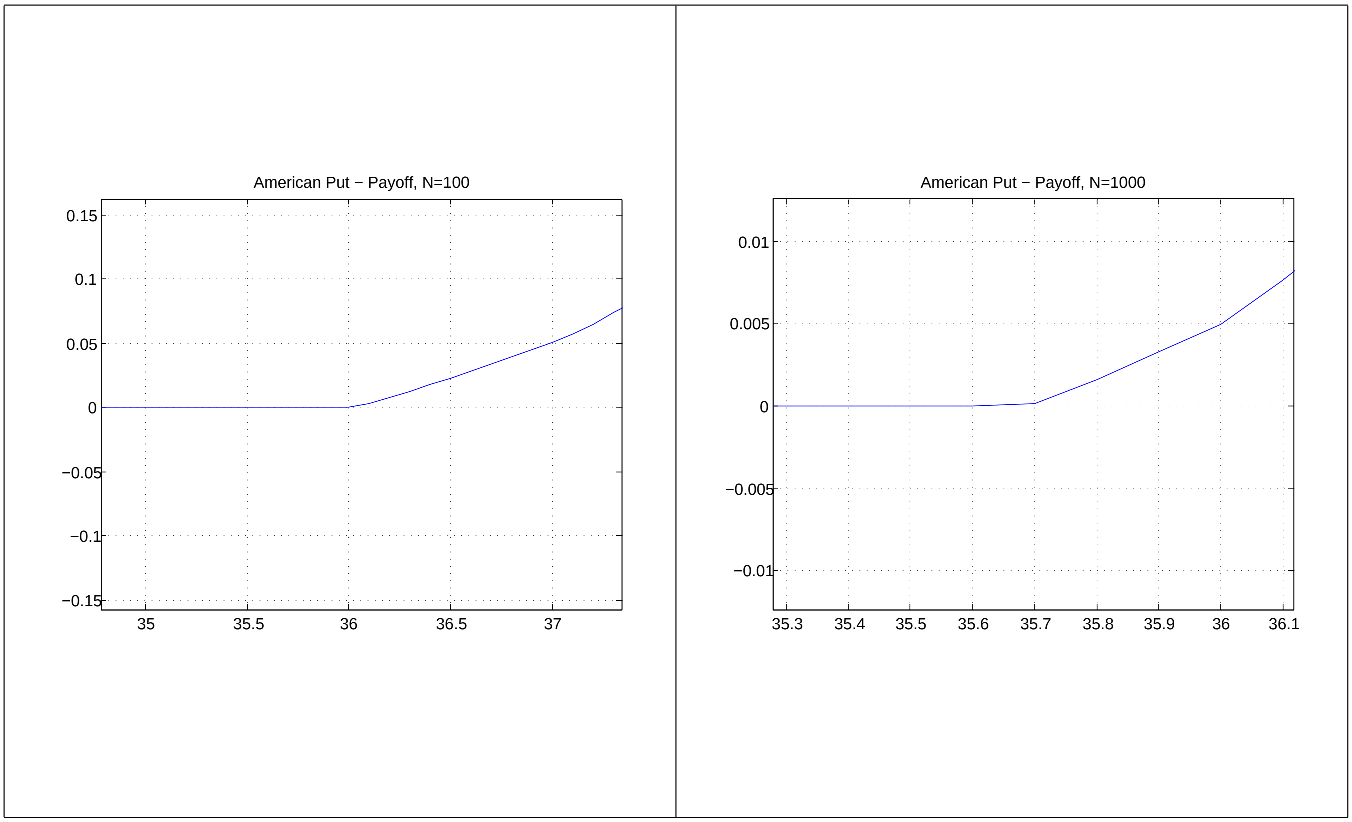
```
ans = 13.811
```

That's pretty close. So $S = 36.2$ as the first point of contact is a pretty good estimate.

Here's another way to do it graphically. Since we're really interested in where the difference between the American put and payoff curve, plot it with Matlab – we've already got the data.

```
figure
plot(S,AP-payoff)
```

Now just just use the zoom feature. Here's what you get with $N = 100$ and the more refined $N = 1000$.



From these pictures, had we used $N = 100$, we would estimate that the touching point is about $S = 36$ and if we had used more precision in calculations, we would estimate that the touching point is about $S = 35.7$.