A REVIEW OF CALCULUS CONCEPTS FOR JDEP 384H

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Contents

Rates of Change and Derivatives

Life would be simple if every function were a constant. It isn't and they aren't. Calculus allows us to get a handle on how functions change with their argument. Recall this definition.

Definition. A function f is a rule of correspondence that assigns to each element x in a set D, called its domain, a unique value $f(x)$ in a set R, called its range (or target). We write $f : D \to R$ in this situation

Example. Let $D = [0, 5]$, the interval of real numbers $x \in \mathbb{R}$ such that $-\pi \leq x \leq \pi$, and define a function $f : D \to \mathbb{R}$ by the formula

$$f(x) = \frac{3}{1 + x^2} + 2x.$$

Certainly this formula yields one and only one well-defined value $f(x)$ for each choice of x. (There is something to check: could the denominator be zero for some x in D? Answer is no.) So we have a function.

This function could be a model of, for example, total cost for a certain product as a function of output x. We are interested in how this function changes with x. A traditional definition of marginal cost says that it is the additional costs incurred by the production of one additional unit of the product. Thus the marginal costs at production level x are

$$f(x+1) - f(x) = \frac{f(x+1) - f(x)}{1} = \frac{f(x+\Delta x) - f(x)}{\Delta x},\ \Delta x = 1.$$

This is just a step away from the calculus definition of marginal cost, namely, the *derivative* of $f(x)$ at x defined by

$$f'(x) = \frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

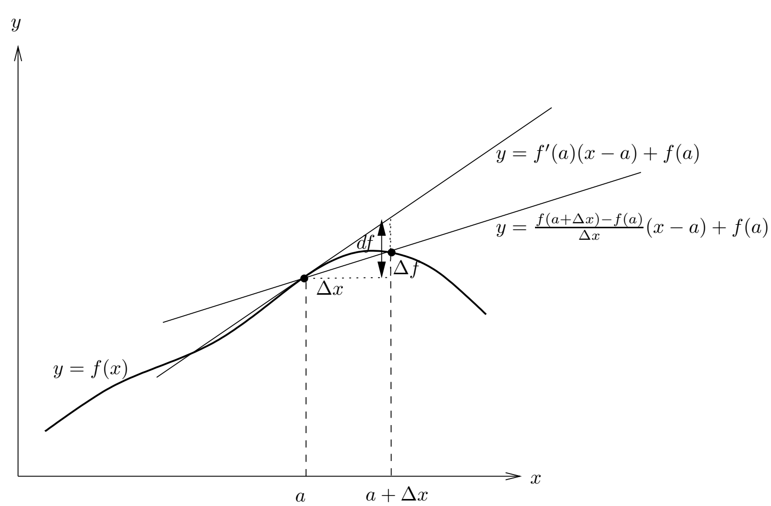
Recall that the derivative has another interpretation, more geometrical in nature, namely, $f'(a)$ is the slope of the tangent line

$$y = f'(a)(x-a) + f(a)$$

to the curve $y = f(x)$ at the point $(a, f(a))$ on the curve. Observe that this tangent line is really a limit of secant lines of the form

$$y = \frac{f(a+h) - f(a)}{h}(x-a) + f(a)$$

where we let $h \to 0$. See the figure below for a comparison.



As we know (and won't give too much detail here) there are many useful rules of differentiation, e.g., for given functions $f(x), g(x)$ and constants a, b, n,

$$\begin{aligned}
(af(x) + bg(x))' &= af'(x) + bg'(x) \\
(f(x)g(x))' &= f(x)g'(x) + f'(x)g(x) \\
\left(\frac{f(x)}{g(x)}\right)' &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\
f(g(x))' &= f'(g(x))g'(x) \\
(x^n)' &= nx^{n-1} \\
(e^x)' &= e^x \\
(\ln(x))' &= \frac{1}{x} \\
(\sin(x))' &= \cos(x) \\
(\cos(x))' &= -\sin(x) \\
(\arctan(x))' &= \frac{1}{1+x^2}
\end{aligned}$$

and so forth. Recall that $F(x)$ is an *antiderivative* of $f(x)$ if $F'(x) = f(x)$. Any two antiderivatives of $f(x)$ differ by a constant, so a general formula for the antiderivatives of $f(x)$ is given by

$$\int f(x)\, dx = F(x) + C,$$

where C is a *constant of integration*. Thus, each derivative formula gives rise to an antiderivative formula. For example, the last derivative formula above implies that

$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C.$$

Example. Find an equation of the tangent line to the curve

$$f(x) = \frac{3}{1+x^2} + 2x$$

at the point on the curve where $x = 3$.

Solution. First use the derivative properties to calculate

$$f'(x) = \left(\frac{3}{1+x^2}\right)' + (2x)' = -\frac{6x}{(1+x^2)^2} + 2.$$

Evaluate and find that $f(3) = 6.3$, $f'(3) = 1.82$, so that the tangent line is given by

$$y = 1.82(x-3) + 6.3 = 1.82x + 0.84.$$

Differentials

Take a closer look at the definition of derivative. We can think of the rate of change over the interval $[a, a+h]$ given by

$$\frac{\Delta f}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

as an approximation to the derivative $f'(a)$. Or, if we are really interested in a subsequent value $f(a+\Delta x)$ beyond a given $f(a)$, we can think of the derivative as giving an approximation to Δf by way of the formula

$$\Delta f \approx df \equiv f'(a)\, dx$$

where we take $\Delta x = dx$. The quantity df defined in the above equality is called the differential of $f(x)$ at $x = a$. In general, we define

$$df = df(x, dx) \equiv f'(x)\, dx.$$

The differential is really a function of the independent variables x and dx.There is a nice geometrical picture that one can draw that shows that we obtain the values df and Δf from the tangent and secant curves at x. For small values of dx the differential provides an excellent approximation to Δf and conversely. Refer to the figure above and identify Δf and df in the picture.

Example. Use the calculations of the previous example to approximate $f(2)$ and $f(4)$ using differentials and the values of f, f'at $x = 3$.

Solution. We obtain that with $dx = 1$,

$$\Delta f = f(3+dx) - f(3) \approx df(3,1) = f'(3)\ 1 = 1.82,$$

so that

$$f(3+1) \approx f(3) + 1.82 = 8.12.$$

As a matter of fact, $f(4) = 4.1765$.

For $dx = -1$, we obtain similarly that

$$\Delta f = f(3+dx) - f(3) \approx df(3,1)\ (-1) = -1.82,$$

so that

$$f(3-1) \approx f(3) - 1.82 = 4.48.$$

As a matter of fact, $f(4) = 8.175$ and $f(2) = 4.2$.

Approximating values with differentials amounts to using a linear approximation to $f(x)$ which is increasingly accurate near $x = a$. The idea is that for x near a,

$$f(x) \approx f(a) + f'(a)(x-a).$$

One can apply this argument to higher derivatives and integrate (next section) to obtain the famous Taylor formula

$$f(x) \approx f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \equiv P_n(x).$$

In fact the error of approximation is well understood. One form of it is

$$R_n(x) \equiv f(x) - P_n(x) = f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!},$$

where ξ is some number between a and x.

Area and Integrals

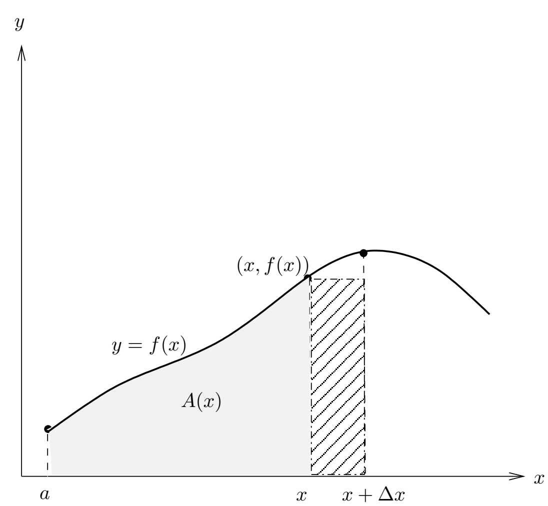
Let $A(x)$ be the signed area between the curve $y = f(x)$ and the x-axis, with vertical line boundaries at $x = a$ and x. Thus $A(a) = 0$. We also write

$$A(x) = \int_a^x f(x)\, dx.$$

This is motivated by the approximate equality

$$A(x + dx) - A(x) \approx f(x)\, dx$$

whose accuracy increases to equality as $dx \to 0$. A graph of the area shows why this is so, so examine the following figure.



If we divide by dx and pass to the limit, we see that

$$\frac{d}{dx}A(x) = f(x).$$

This is one form of the fundamental theorem of calculus (FTOC). The other form follows from this argument: As we saw earlier, any two antiderivatives of $f(x)$ differ by a constant, so if $F'(x) = f(x)$, then $F(x) + C = A(x)$, where C is some constant. It follows that

$$\int_a^b f(x)\ dx = A(b) = A(b) - A(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a),$$

which gives the second form of FTOC: If $F'(x) = f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a) \equiv F(x)|_{x=a}^{b}$$

Example. Let $f(x)$be as in the first example, and calculate $\int_0^4 f(x)\ dx$.

Solution. Here we have to find an antiderivative $f(x)$ which we write in the customary indefinite integral form $F(x) = \int f(x)\ dx$. We leave it to the reader to check that

$$\int \left(\frac{3}{1+x^2} + 2x\right) dx = 3\int \frac{dx}{1+x^2} + 2\int x\,dx = 3\arctan(x) + 2\frac{x^2}{2} + C$$

where C is an arbitrary constant of integration. From this we deduce that

$$\int_0^4 \left(\frac{3}{1+x^2} + 2x\right) dx = 3\arctan(x) + x^2\Big|_{x=0}^{4} \approx 19.977.$$

Perhaps the first form of the definite integral that you saw was as a limit of Riemann sums, of which the following is a special case: Let $a = x_0 < x_1 \cdots < x_N = b$ with $x_{j+1} - x_j = \Delta x \equiv \frac{b-a}{N}$

$$\int_a^b f(x)\ dx = \lim_{\Delta x \to 0} \sum_{j=0}^{N-1} f(x_j)\,\Delta x.$$

Of course, this again gives us the signed area between $y = f(x)$ and the x-axis. This is a left Riemann sum. The corresponding right Riemann sum is $\sum_{j=1}^{N} f(x_j)\,\Delta x$. A rather fortunate turn of events occurs when we average the left and right Riemann sums for a given N.

Multivariate Calculus

Life would be simpler if all functions involved one variable. They don't. For example, the volume of a right circular cylinder of radius x and height y is a function of two variables

$$f(x, y) = \pi x^2 y.$$

Here we understand that the domain of the function V consists of points (x, y) such that x and y are non-negative. How do we make sense of rates of change of such variables when there are two (or more!) independent variables. The answer is that we take derivatives with respect to each independent variable separately, treating all other variables as constant, and use our single variable rules. Such derivatives are *partial derivatives* since they only tell us part of the rate of change information about f. Thus, in our example,

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 2\pi xy \\ \frac{\partial f}{\partial y}(x,y) &= \pi x^2. \end{aligned}$$

There is a nice interpretation of these derivatives as simply ordinary derivatives of functions of one variable obtained by intersection the surface $z = f(x,y)$ with vertical planes parallel to the x- or y-axes.

Similarly, there are higher analogues of integrals and differentials. We won't go into detail here, but in a nutshell, the double integral over a region R in the xy-plane of a continuous function $f(x,y)$ defined on that region is a number

$$\iint_R f(x,y)\; dA$$

that represents the signed volume between the graph of $z = f(x,y)$, $(x,y) \in R$ and the xy-plane with vertical sides along the boundary of R.

Finally, there is the important idea of differentials for functions of more than one variable. Just as differentials represent tangent line approximations to a curve $y = f(x)$, differentials for a function of two variables represent tangent plane approximations to a surface $z = f(x,y)$. Here is the definition of differential for a function $f(x,y)$ of two variables with continuous partial derivative:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

This definition is completely analogous to differentials in one variable. It should be noted that df is really a function of the four independent variables x, y, dx and dy.

Just for the record, the definition above gives rise to a kind of chain rule for certain functions of two arguments. Suppose that we know that $x = x(t)$ and $y = y(t)$ are both functions of t, so that $f = f(x(t), y(t))$is really a function of the single independent variable t. Then what is df/dt? The answer is a *chain rule* for a function of t that has two intermediate variables x, y:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example. Given that $f(x, y) = x^2y + y^3 + x^2$, find a formula for the differential df and in particular, for the differential evaluated at $x = 2$, $y = 1$. How does this help you describe the tangent plane approximation to $z = f(x, y)$ for (x, y) near the point $(2, 1)$?

Solution. In this case, $\partial f/\partial x = 2xy + 2x$ and $\partial f/\partial y = x^2 + 3y^2$. Therefore, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (2xy + 2x)\,dx + \left(x^2 + 3y^2\right)dy$$

so that

$$df\,(2, 3, dx, dy) = 8dx + 7dy.$$

In particular, if we take $dx = x - 2$ and $dy = 1$, then we obtain the expression

$$df = 8\,(x - 2) + 7\,(y - 1)$$

and if we interpret $df \approx \Delta f$ for points (x, y) near $(2, 1)$, then we obtain the expression $f(x, y) \approx z$, where

$$z - f(2, 1) = z - 9 = 8\,(x - 2) + 7\,(y - 1)\,,$$

that is,

$$z = 8x + 7y - 14,$$

which is a plane containing the point $(2, 1, f(2, 1))$ and is in fact the equation of the tangent plane to the surface at $(2, 1, 9)$.