(20) 1. The system of equations
\[
\begin{align*}
    x_1 + x_2 &= 2 \\
    x_3 + x_4 &= 1 \\
    2x_1 + 4x_2 + 2x_4 &= 6
\end{align*}
\]
has coefficient matrix $A$ and right hand side $b$ such that the row-reduced echelon form of $[A|b]$ is
\[
\begin{bmatrix}
    1 & 2 & 0 & 0 & | & 2 \\
    0 & 0 & 1 & 0 & | & 0 \\
    0 & 0 & 0 & 1 & | & 1
\end{bmatrix}
\]. Use this information to answer the following:
(a) Find a basis for the null space of $A$.
(b) Find the form of a general solution of the system $Ax = b$.
(c) Find a basis for the row space of $A$.
(d) No matter what the right hand side of $b$ is, this system has solutions. In terms of rank, why do we know this?

(12) 2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.
(a) Find $A^{-1}$.
(b) Use (a) to solve the equation $Ax = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ for $x$

(22) 3. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$.
(a) Find the reduced row-echelon form and rank of $A$.
(b) Find a basis for the column space of $A$.
(c) Determine which of the following vectors is in the column space of $A$ and, if so, express the vector as a linear combination of the columns of $A$:
$b_1 = [2, 1, 0]^T$, $b_2 = [2, -3, 3]^T$.

(14) 4. Use the Gram-Schmidt process on the basis
$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ of the subspace $W$ of $\mathbb{R}^4$ to produce an orthonormal basis of $W$. 
5. Calculate the following determinant:
\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 \\
1 & 3 & 5 & 8 \\
1 & 4 & 9 & 9
\end{vmatrix}
\]

6. The set \( \{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\} \) is an orthogonal basis of the inner product space \( P_2 \) of polynomials of degree at most 2 with the inner product \( \langle p, q \rangle = \int_0^1 p(x) q(x) \, dx \). Assume this and find the coordinates of \( p(x) = x^2 \) with respect to this basis. What is the angle between \( p(x) \) and \( q(x) = 1 \) in this inner product space?

7. Find a least squares solution to the inconsistent system
\[
\begin{bmatrix}
1 & -1 & 1 & 1 \\
1 & 1 & 2 & 1 \\
1 & 2 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
3
\end{bmatrix}
\]

8. Let \( A = \begin{bmatrix}
2 & 1 & 1 \\
0 & 2 & 1 \\
0 & 1 & 2
\end{bmatrix} \).
(a) Find all eigenvalues of \( A \).
(b) Find a basis for the eigenspace corresponding to each eigenvalue of \( A \).
(c) Produce an invertible matrix \( P \) and diagonal \( D \) such that \( P^{-1}AP = D \).

9. Let \( A = \begin{bmatrix}
1 & 1 + i \\
1 - i & 2
\end{bmatrix} \). One of the eigenvalues of \( A \) is 0.
(a) Find the eigenvalues of \( A \).
(b) Find a basis for the eigenspace corresponding to each eigenvalue of \( A \).
(c) Produce a unitary matrix \( U \) and diagonal matrix \( D \) such that \( U^*AU = D \).

10. Let \( v \) be a unit column vector in \( \mathbb{R}^n \) and \( H = I_n - 2vv^T \).
(a) What is the size of the matrix \( H \)?
(b) Give the definition of symmetric matrix and prove \( H \) is symmetric.
(c) Give the definition of orthogonal and prove \( H \) is orthogonal.

11. Circle T for true, F for false or do not answer. Each correct answer is worth 3 points, incorrect answer worth -1 points and no answer worth 0, for a minimum of 0 and maximum of 21 points.
T F (a) If \( u \) and \( v \) are elements of the real inner product space \( V \), then \( \langle u, v \rangle < \langle v, v \rangle \geq \langle u, v \rangle^2 \).
T F (b) Every real matrix is similar to a diagonal matrix.
T F (c) Every orthogonal set of vectors is linearly independent
T F (e) For \( n \times n \) matrices \( A \) and \( B \), \( (AB)^* = A^*B^* \).
T F (f) If the linear system \( Ax = B \) has a unique solution and \( A \) is an \( m \times n \) matrix, then \( n \leq m \).
T F (g) If \( V = \text{span} \{v_1, v_2, v_3\} \) and \( \dim(V) = 2 \), then \( \{v_1, v_2, v_3\} \) is a linearly dependent set.

12. Fill in the blank or give a short answer in the following:
(a) \[
\begin{bmatrix}
1 & 2 \\
0 & 1 + i \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 0 \\
0 & i & 1
\end{bmatrix} =
\]

(b) A set of vectors \( v_1, v_2, \ldots, v_n \) in the vector space \( V \) is defined to be linearly independent if:

(c) If \( A \) is symmetric matrix, what can you say about the eigenvalues of \( A \)?

(d) Find two \( 2 \times 2 \) matrices \( A \) and \( B \) such that \( AB = 0 \).

(f) Let \( u = [1, 2, 1]^T \) and \( v = [1, -1, 0] \). Then the projection of \( u \) along \( v \) is \( p \) and \( u \) can be written as \( p + x \) where \( x \) is orthogonal to \( v \). Find \( p, x \).

(g) If \( W \) is the row space of the matrix \[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]
then \( W^\perp = \).

(h) If \( u = (-2, 1 + i, 3, 1) \in \mathbb{C}^4 \), then \( \|u\|_1 \) and \( \|u\|_\infty \) are equal to: