

Name: \_\_\_\_\_

Score: \_\_\_\_\_

*Instructions:* Show your work in the spaces provided below for full credit. Use the reverse side for additional space, *but clearly so indicate*. You must clearly identify answers and show supporting work to receive any credit. Exact answers (e.g.,  $\pi$ ) are preferred to inexact (e.g., 3.14). Point values of problems are given in parentheses. Notes or text in *any* form not allowed. Calculator is allowed.

(20) **1.** The system of equations

$$\begin{aligned}x_1 + x_2 &= 2 \\x_3 + x_4 &= 1 \\2x_1 + 4x_2 + 2x_4 &= 6\end{aligned}$$

has coefficient matrix  $A$  and right hand side  $\mathbf{b}$  such that the row-reduced echelon form of  $[A|\mathbf{b}]$  is

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right].$$
 Use this information to answer the following:

- (a) Find a basis for the null space of  $A$ .  
 (b) Find the form of a general solution of the system  $A\mathbf{x} = \mathbf{b}$ .  
 (c) Find a basis for the row space of  $A$ .  
 (d) No matter what the right hand side of  $\mathbf{b}$  is, this system has solutions. In terms of rank, why do we know this?

(12) **2.** Let  $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$ .

(a) Find  $A^{-1}$ .(b) Use (a) to solve the equation  $A\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$  for  $\mathbf{x}$ 

(22) **3.** Let  $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$ .

- (a) Find the reduced row-echelon form and rank of  $A$ .  
 (b) Find a basis for the column space of  $A$ .  
 (c) Determine which of the following vectors is in the column space of  $A$  and, if so, express the vector as a linear combination of the columns of  $A$ :  
 $b_1 = [2, 1, 0]^T$ ,  $b_2 = [2, -3, 3]^T$ .

(14) **4.** Use the Gram-Schmidt process on the basis

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ , and  $u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  of the subspace  $W$  of  $\mathbb{R}^4$  to produce an orthonormal basis of  $W$ .

(12) **5.** Calculate the following determinant:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 8 \\ 1 & 4 & 9 & 9 \end{vmatrix}$$

(12) **6.** The set  $\{1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}\}$  is an orthogonal basis of the inner product space  $\mathcal{P}_2$  of polynomials of degree at most 2 with the inner product  $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ . Assume this and find the coordinates of  $p(x) = x^2$  with respect to this basis. What is the angle between  $p(x)$  and  $q(x) = 1$  in this inner product space?

(12) **7.** Find a least squares solution to the inconsistent system

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

(16) **8.** Let  $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ .

(a) Find all eigenvalues of  $A$ .

(b) Find a basis for the eigenspace corresponding to each eigenvalue of  $A$ .

(c) Produce an invertible matrix  $P$  and diagonal  $D$  such that  $P^{-1}AP = D$ .

(16) **9.** Let  $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ . One of the eigenvalues of  $A$  is 0.

(a) Find the eigenvalues of  $A$ .

(b) Find a basis for the eigenspace corresponding to each eigenvalue of  $A$ .

(c) Produce a unitary matrix  $U$  and diagonal matrix  $D$  such that  $U^*AU = D$ .

(15) **10.** Let  $v$  be a unit column vector in  $\mathbb{R}^n$  and  $H = I_n - 2vv^T$ .

(a) What is the size of the matrix  $H$ ?

(b) Give the definition of symmetric matrix and prove  $H$  is symmetric.

(c) Give the definition of orthogonal and prove  $H$  is orthogonal.

(21) **11.** Circle T for true, F for false or do not answer. Each correct answer is worth 3 points, incorrect answer worth -1 points and no answer worth 0, for a minimum of 0 and maximum of 21 points.

T F (a) If  $\mathbf{u}$  and  $\mathbf{v}$  are elements of the real inner product space  $V$ , then  $\langle \mathbf{u}, \mathbf{v} \rangle < \langle \mathbf{v}, \mathbf{v} \rangle \geq \langle \mathbf{u}, \mathbf{v} \rangle^2$ .

T F (b) Every real matrix is similar to a diagonal matrix.

T F (c) Every orthogonal set of vectors is linearly independent

T F (e) For  $n \times n$  matrices  $A$  and  $B$ ,  $(AB)^* = A^*B^*$ .

T F (f) If the linear system  $Ax = B$  has a unique solution and  $A$  is an  $m \times n$  matrix, then  $n \leq m$ .

T F (g) If  $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  and  $\dim(V) = 2$ , then  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a linearly dependent set.

(28) **12.** Fill in the blank or give a short answer in the following:

(a) 
$$\begin{bmatrix} 1 & 2 \\ 0 & 1+i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & i & 1 \end{bmatrix} =$$

(b) A set of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  in the vector space  $V$  is defined to be linearly independent if:

(c) If  $A$  is symmetric matrix, what can you say about the eigenvalues of  $A$ ?

(d) Find two  $2 \times 2$  matrices  $A$  and  $B$  such that  $AB = 0$ .

(f) Let  $\mathbf{u} = [1, 2, 1]^T$  and  $\mathbf{v} = [1, -1, 0]$ . Then the projection of  $\mathbf{u}$  along  $\mathbf{v}$  is  $\mathbf{p}$  and  $\mathbf{u}$  can be written as  $\mathbf{p} + \mathbf{x}$  where  $\mathbf{x}$  is orthogonal to  $\mathbf{v}$ . Find  $\mathbf{p}, \mathbf{x}$ .

(g) If  $W$  is the row space of the matrix  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$  then  $W^\perp =$ .

(h) If  $\mathbf{u} = (-2, 1 + i, 3, 1) \in \mathbb{C}^4$ , then  $\|\mathbf{u}\|_1$  and  $\|\mathbf{u}\|_\infty$  are equal to: