(30) 1. Let \( A = [v_1, v_2, v_3, v_4] = \begin{bmatrix} 1 & 4 & 1 & -1 \\ 2 & 4 & 1 & -1 \\ 4 & 8 & 2 & -2 \end{bmatrix} \) with reduced row echelon form \( R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Find a basis for \( \mathcal{R}(A) \), the row space of \( A \).

(b) Find a basis for \( \mathcal{C}(A) \), the column space of \( A \).

(c) Find a basis for \( \mathcal{N}(A) \), the null space of \( A \).

(d) Find all possible linear combinations of the vectors \( v_1, v_2, v_3, v_4 \) that sum to 0.

(e) Which \( v_j \)'s are redundant in the list of vectors \( v_1, v_2, v_3, v_4 \)?

(f) Find a basis of \( \mathbb{R}^3 \) containing a basis of \( \mathcal{C}(A) \).
(16) 2. Use the Subspace Test to decide if \( W \) is a subspace of the vector space \( V \), where

(a) \( V = \mathbb{R}^3 \) and \( W = \{(a, b, a - b + 1) \mid a, b \in \mathbb{R}\} \)

(b) \( V = C[0, 1] \), the continuous functions on \([0, 1]\) and \( W = \{f(x) \mid f(x) \in C[0, 1] \text{ and } f(1) = 0\} \).

(10) 3. Assume that \(1 + x, x + x^2, 1 - x\) is a basis of \( \mathcal{P}_2 \), the space of polynomials of degree at most two, and find the coordinates of \( 2 + x^2 \) relative to this basis.
(8) 4. Let \( A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \). Find the adjoint matrix \( \text{adj}(A) \) of \( A \).

(12) 5. You are given that \( w_1 = (0, 1, 0), \ w_2 = (1, 1, 1) \) is a linearly independent set in \( V = \mathbb{R}^3 \) and \( v_1 = (1, 3, 1), \ v_2 = (2, -1, 1), \ v_3 = (1, 0, 1) \) is a basis of \( V \). Steinitz substitution says that \( w_1, w_2 \) can be substituted into the basis in place of certain \( v_i \)'s. Which substitutions work?
6. Fill in the blanks or answer True/False (T/F).

(a) Every vector space is finite dimensional (T/F) ______________

(b) Elementary row operations on a matrix do not change the column space (T/F) ______________

(c) If \( x = x_0 \) and \( x = x_1 \) are both vector solutions to the linear system \( Ax = b \), then \( x_1 - x_0 \) is in the null space of \( A \). (T/F) ______________.

(d) The function \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( T((x, y)) = (x + y, x - 2y) \) is linear (T/F) ______________ and one-to-one (T/F) ______________.

(e) The Basis Theorem asserts that every finite dimensional vector space ________________________.

(f) The Dimension Theorem asserts that ________________________________.

(g) A basis of a vector space is by definition ________________________________.

7. (a) Show that the columns of the matrix
\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 1 \\
3 & 0 & 1
\end{bmatrix}
\]
form a linearly dependent set.

(b) (Honors students only) Prove that any set of vectors \( v_1, v_2, \ldots, v_n \) in a vector space \( V \) that contains the zero vector is a linearly dependent set.