

ON RINGS FOR WHICH FINITELY GENERATED IDEALS HAVE ONLY FINITELY MANY MINIMAL COMPONENTS

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ABSTRACT. For a commutative ring R we investigate the property that the sets of minimal primes of finitely generated ideals of R are always finite. We prove this property passes to polynomial ring extensions (in an arbitrary number of variables) over R as well as to R -algebras which are finitely presented as R -modules.

1. INTRODUCTION

In [OP] Ohm and Pendleton examine several topological properties which may be possessed by the prime spectrum of a commutative ring R . One of these properties, which Ohm and Pendleton call *FC* for ‘finite components’, is that every closed subset of $\text{Spec } R$ has a finite number of irreducible components, or equivalently, that every quotient R/I has a finite number of minimal primes. In this paper, we will call a ring R such that $\text{Spec } R$ has property FC an *FC-ring* or simply say R is *FC*. Such a condition on R can be useful when investigating questions concerning heights of prime ideals. For example, using prime avoidance one can show that if P is a prime ideal of an FC-ring of height at least h then P contains an ideal generated by h elements which has height at least h . And if R is a quasi-local FC-ring of dimension d then every radical ideal is the radical of a d -generated ideal.

An obvious question is whether the FC property passes to finitely generated algebras. Heinzer [He] showed that if R is an FC-ring and S is a finite R -algebra then S is FC, while Ohm and Pendleton prove that if R is an FC-ring and $\dim R$ is finite then $R[x]$ is FC. However, if R is an infinite-dimensional FC-ring $R[x]$ need not be FC. This is illustrated by the following example:

Example 1.1. ([OP, Example 2.9]) Let V be a valuation domain of countably infinite rank and let $\text{Spec } R = \{P_i \mid i \in \mathbb{N}_0\}$ where $P_i \subset P_{i+1}$ for all i . Clearly, V is an FC-ring. For each $i \in \mathbb{N}_0$ let $a_i \in P_{i+1} \setminus P_i$. Let x be an indeterminate and set

$$f_i = a_i \prod_{j=0}^i (a_j x - 1) \in V[x]$$

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for each $i \in \mathbb{N}_0$. Then the ideal $I = (\{f_i \mid i \in \mathbb{N}_0\})V[x]$ has infinitely many minimal primes (namely, $(P_i, a_i x - 1)V[x]$ for $i \in \mathbb{N}_0$). Hence, $V[x]$ is not an FC-ring.

In this paper we examine the weaker property that every *finitely generated* ideal of the ring has finitely many minimal primes. In many applications of FC (such as the ones mentioned in the first paragraph above), one only needs the FC property for finitely generated ideals. We show that this weaker property passes to polynomial extensions (in any number of variables) as well as to finitely presented algebras.

This paper arose in connection to the author's work on the Cohen-Macaulay property for non-Noetherian rings [HM]. It is an open question whether the Cohen-Macaulay property (as defined in [HM]) passes to polynomial extensions, but there is reason to believe the answer is 'yes' if finitely generated ideals in the base ring have only finitely many minimal primes.

2. MAIN RESULTS

All rings in this paper are assumed to be commutative with identity. For a ring R and a module M we let $\text{Supp}_R M$ denote the support of M and $\text{Min}_R M$ denote the minimal elements of $\text{Supp}_R M$. We begin with the following definition:

Definition 2.1. Let R be a ring. An R -module M is said to be *FGFC* if for every finitely generated R -submodule N of M it holds that $\text{Supp}_R M/N$ is the finite union of irreducible closed subsets of $\text{Spec } R$; equivalently, $\text{Min}_R M/N$ is a finite set. A ring R is an *FGFC-ring* if it is FGFC as an R -module.

We make some elementary observations:

Proposition 2.2. *Let R be an FGFC ring.*

- (a) *Every finitely presented R -module is FGFC.*
- (b) *If I is a finitely generated ideal then R/I is FGFC.*
- (c) *R_S is an FGFC ring for every multiplicatively closed set S of R .*
- (d) *Every prime p of finite height is the radical of a finitely generated ideal.*
- (e) *R/p is an FGFC ring for every minimal prime p of R .*

Proof: Note that if M is finitely presented then $\text{Min}_R M = \text{Min}_R R/F_0(M)$, where $F_0(M)$ is the zeroth Fitting ideal of M ([E, Proposition 20.7]). Thus, $\text{Min}_R M$ is finite. Since every quotient of a finitely presented R -module by a finitely generated submodule is also finitely presented, this proves (a). Parts (b) and (c) are clear. Part (d) is proved by induction and prime avoidance. Part (e) follows from (b) and (d). \square

Clearly, every FC-ring is FGFC. However, the converse is not true. The ring $V[x]$ of Example 1.1 is not FC but is FGFC by the following theorem:

Theorem 2.3. *Let R be a ring. Then R is an FGFC-ring if and only if $R[x]$ is.*

Proof: Since $R \cong R[x]/(x)$ it is clear that R is FGFC if $R[x]$ is. Assume that R is FGFC. For a finitely generated ideal I of $R[x]$ let $d(I) := \min\{\sum_{i=1}^n \deg f_i \mid I = (f_1, \dots, f_n)\}$. If $d(I) = 0$ then $I = JR[x]$ where J is a finitely generated ideal

of R . Hence $\text{Min}_{R[x]} R[x]/I = \{pR[x] \mid p \in \text{Min}_R R/J\}$, which is a finite set as R is FGFC. Suppose I is a finitely generated ideal of $R[x]$ with $d(I) > 0$ and the theorem holds for all finitely generated ideals J of $R[x]$ such that $d(J) < d(I)$. Let f_1, \dots, f_n be a set of generators for I such that $\sum_{i=1}^n \deg f_i = d(I)$. Without loss of generality we may assume $\deg f_n = \min_{1 \leq i \leq n} \{\deg f_i \mid \deg f_i > 0\}$. Let c be the leading coefficient of f_n and Q a minimal prime containing I . If $c \in Q$ then Q is minimal over $I' = (f_1, \dots, f_{n-1}, f_n - cx^{\deg f_n}, c)$. As $d(I') < d(I)$ we have that $\text{Min}_{R[x]} R[x]/I'$ is a finite set. Suppose that $c \notin Q$. Then Q_c is minimal over I_c . We claim that $\text{Min}_{R_c[x]} R_c[x]/I_c$ is finite.

Case 1: $\deg f_i \geq \deg f_n$ for some $i < n$.

In this case, since f_n is monic in $R_c[x]$ we can replace the generator f_i by one of smaller degree (using f_n). Hence, $d(I_c) < d(I)$ and $\text{Min}_{R_c[x]} R_c[x]/I$ is a finite set.

Case 2: $\deg f_i < \deg f_n$ for all $i < n$.

By assumption on $\deg f_n$ we have f_1, \dots, f_{n-1} have degree zero. By replacing R with $R_c/(f_1, \dots, f_{n-1})R_c$ (which is still FGFC), we can reduce to the case $I = (f(x))$ where $f(x)$ is a monic polynomial of positive degree. As $S = R[x]/(f(x))$ is a free R -module, the going-down theorem holds between S and R . Therefore, every minimal prime of S contracts to a minimal prime of R . Further, since S is finite as an R -module, there are only finitely many primes of S contracting to a given prime of R . Hence, $\text{Min}_{R[x]} R[x]/(f(x))$ is finite. \square

As a consequence, we get the following:

Corollary 2.4. *Let R be an FGFC-ring and X a (possibly infinite) set of indeterminates over R . Then $R[X]$ is an FGFC-ring.*

Proof: Let $I = (f_1, \dots, f_n)$ be a finitely generated ideal of $R[X]$. Then there exists $x_1, \dots, x_m \in X$ such that $f_i \in R[x_1, \dots, x_m]$ for all i . Let $S = R[x_1, \dots, x_m]$, $J = (f_1, \dots, f_n)S$ and $X' = X \setminus \{x_1, \dots, x_m\}$. By the theorem (and induction) we have $\text{Min}_S S/J$ is finite. Furthermore, since $I = JS[X']$, every prime minimal over I is of the form $pS[X']$ for some $p \in \text{Min}_S S/J$. \square

It should be clear that, unlike FC, the FGFC property does not pass to arbitrary finite ring extensions. For example, if $V[x]$ and I are as in Example 1.1, then $V[x]$ is FGFC but $V[x]/I$ is not. However, FGFC does pass to finitely presented algebras:

Corollary 2.5. *Let R be an FGFC-ring. Then any R -algebra which is finitely presented as an R -module is an FGFC-ring.*

Proof: Let S be an R -algebra which is finitely presented over R . Then certainly $S \cong R[x_1, \dots, x_n]/I$ for some ideal I of $R[x_1, \dots, x_n]$. By the theorem it is enough to show that I is finitely generated. Since S is a finite R -module, for each $i = 1, \dots, n$ there exists a monic polynomial $f_i(t) \in R[t]$ such that $f_i(x_i) \in I$. Let $J = (f_1(x_1), \dots, f_n(x_n))R[x_1, \dots, x_n]$, $L = I/J$, and $T = R[x_1, \dots, x_n]/J$. It suffices to show that L is a finitely generated ideal of T . In fact, since T is a free R -module of finite rank and T/L is finitely presented over R , it follows by the snake lemma that L is finitely generated as an R -module. \square

Given an integral ring extension $R \subset S$, it is clear that if S is FGFC then so is R . For, any prime minimal over a finitely generated ideal I of R is contracted from a prime of S minimal over IS . However, FGFC does not in general ascend from R to S even if R is coherent and S is finite over R . To see this, one can again modify the example of Ohm and Pendleton:

Example 2.6. Let $R = V[x]$ and I be as in Example 1.1. Let $S = R[y]/(yI, y^2 - y)$. Then R is a coherent domain (cf. [GV]), $R \subset S$ and S is a finite R -module. However, R is FGFC but S is not.

As a final result, we prove that FGFC does ascend from coherent domains to finite torsion-free algebras:

Proposition 2.7. *Suppose R is a coherent domain and S a finite R -algebra which is torsion-free as an R -module. If R is FGFC then so is S .*

Proof: The hypotheses imply that S is isomorphic as an R -module to a finitely generated submodule of R^n for some n . As R is coherent, S is finitely presented as an R -module. By Corollary 2.5, S is an FGFC ring. \square

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