Math 918

Homework # 2

1. (Katie) Let R be a Noetherian local ring of dimension d and I an ideal of R. Prove that $\operatorname{codim} I \geq i$ if and only if I contains $x_1, \ldots, x_i$ which form part of a system of parameters for R.

2. (Justin) Let (R, m) be a Noetherian local ring and $\mathbf{F}$ a complex

$$0 \to F_s \to F_{s-1} \to \cdots \to F_0 \to 0$$

consisting of finitely generated free modules in each degree and such that all the homology has finite length. Let M be an R-module and $J_i := \operatorname{Ann}_R H^i_m(M)$ for $i \geq 0$. Prove that for each $i \geq 0$, $J_0 \cdot J_1 \cdots J_{s-i}$ annihilates $H_i(\mathbf{F} \otimes_R M)$.

3. (Nick) Let (R, m) be a Cohen-Macaulay ring and $x_1, \ldots, x_d$ a system of parameters for R. Prove that for any positive integers $n_1, \ldots, n_d$,

$$\lambda(R/(x_1^{n_1}, \ldots, x_d^{n_d}) = (\prod_{i=1}^{d} n_i)\lambda(R/(x_1, \ldots, x_d)).$$

4. (Laura) Let (R, m) be a regular local ring of characteristic $p > 0$ and M an R-module of finite length. Prove that $\lambda(F(M)) = p^d \lambda(M)$, where $d = \dim R$. (Hint: Use induction on the length and the previous exercise.)

5. (Lori) Let R be a regular local ring and I an ideal of R. Prove that $F(H^i_I(R)) \cong H^i_I(R)$ for all i.

6. (Brian) Let $\phi : R \to S$ be a homomorphism of commutative rings. For an R-module M, let $S \otimes_\phi M$ denote the left S-module $S \otimes_R M$ where S is viewed as a right R-module via ϕ (i.e., $s \otimes rm = s\phi(r) \otimes m$). In this context, if $\phi : R \to R$ is the Frobenius map, then $R \otimes_\phi M$ is $F(M)$, the Frobenius functor applied to M. By the associative property of tensor products, if $\phi : R \to S$ and $\psi : S \to T$ are ring homomorphisms, then $T \otimes_\psi (S \otimes_\phi M) \cong T \otimes_{\psi\phi} M$. Use this approach to show that Frobenius commutes with localization and completion.

7. (Xuan) Let (R, m) be a local ring of characteristic $p > 0$ and M a finitely generated R-module such that $M \cong F(M)$. Prove that M is free.

8. (Silvia) Let R be a ring of characteristic $p > 0$ and W a multiplicatively closed set of R. Prove that $F(R_S) \cong R_S$. More generally, let M be a flat R-module. Prove that $F(M)$ is flat. (Hint: Use Lazard's Theorem.) (Question: Is $F(M) \cong M$?)

9. (Hamid) Let R be a Noetherian ring of characteristic $p > 0$. Prove that the Frobenius functor is faithful; i.e., $F(M) = 0$ if and only if $M = 0$.