Math 918

Homework # 1

1. (Justin) Prove the monomial conjecture for Cohen-Macaulay local rings.

2. (Hamid) Let (R, m) be a quasi-local ring. Let M be an R-module and suppose $\mathbf{F}$ and $\mathbf{G}$ are two free resolutions of M consisting of finitely generated free R-modules. Suppose $\mathbf{F}$ is minimal (i.e., $\partial(F_i) \subseteq mF_{i-1}$ for all i, where ∂ is the differential of $\mathbf{F}$). Prove that there exists an exact complex $\mathbf{H}$ of finitely generated free R-modules such that $\mathbf{G} \cong \mathbf{F} \oplus \mathbf{H}$ as complexes.

3. (Laura) Let M be a finitely presented R-module and $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ and $G_1 \xrightarrow{\psi} G_0 \to M \to 0$ two presentations of M. Let $r = \operatorname{rank} F_0$ and $s = \operatorname{rank} G_0$. Prove that $I_{r-i}(\phi) = I_{s-i}(\psi)$ for all i. (Hint: It is enough to show the ideals are equal locally, so one may assume R is local. In this case, one can compare any presentation to a fixed minimal presentation, so we can assume the first presentation above is minimal, the second arbitrary. Using exercise 2 above, one can choose bases for G_0 and G_1 so that ψ has the form
$$\begin{pmatrix} \phi & 0 & 0 \\ 0 & I_p & 0 \end{pmatrix}$$
where I_p is the $p \times p$ identity matrix ($p = s - r$).)

4. (Xuan) Let R be a ring and M a finitely presented R-module. Let $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ be a presentation for M. Prove that M is projective if and only if $I_j(\phi)$ is generated by an idempotent for each j.

5. (Brian) Let A be an $n \times m$ matrix with entries from a commutative ring R. Prove that the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if there a nonzero element $z \in R$ such that $zI_m(A) = 0$. (This is a theorem due to McCoy.)

6. (Katie) Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation, and $r = \operatorname{rank} F_0$. Prove that $I_r(\phi) \subseteq \operatorname{Ann}_R M$.

7. (Lori) Let R be a ring, $F_1 \xrightarrow{\phi} F_0 \to M \to 0$ a presentation, and $r = \operatorname{rank} F_0$. Prove that $(\mathrm{Ann_R M})I_j(\phi) \subseteq I_{j+1}(\phi)$. Conclude that $(\mathrm{Ann_R M})^r \subseteq I_r(\phi)$.

8. (Silvia) Let R be a semi-local ring and P a finitely generated projective R-module. Prove that P is free if and only if for all maximal ideals m and n of R, $\operatorname{rank}_{R_m} P_m = \operatorname{rank}_{R_n} P_n$. (For a hint, see Exercise 1.2.27 of Bruns and Herzog.)

9. (Nick) Let R be a ring, M an R-module, and x an indeterminate over R. Suppose $f(x) \in R[x]$ is a zero-divisor on $M[x] = M \otimes_R R[x]$. Prove there exists a nonzero element $u \in M$ such that $f(x)u = 0$ (that is, all the coefficients of f annihilate u).