Math 902
Homework # 4
Due: Friday, March 4th

Except in Problem # 5, \( R \) denotes a commutative ring with identity.

1. Let \( M \) be a flat \( R \)-module. Prove the following conditions are equivalent. (If \( M \) satisfies either of these conditions, \( M \) is said to be \textit{faithfully flat}.)
   
   (a) For every nonzero \( R \)-module \( N \) we have \( M \otimes_R N \neq 0 \).
   (b) For every maximal ideal \( m \) of \( R \) we have \( M \neq mM \).

2. Let \( N \) be an \( R \)-module. Prove that \( \text{Hom}_R(\cdot, N) \) is left exact.

3. Let \( T \) be a (commutative) \( R \)-algebra and \( M \) an \( R \)-module.
   
   (a) Prove that if \( M \) is projective, \( T \otimes_R M \) is a projective \( T \)-module.
   (b) Prove that if \( M \) is flat, \( T \otimes_R M \) is a flat \( T \)-module.

4. In the context of the Five-Lemma, prove that the middle map is surjective.

5. Let \( R \) be a (not necessarily commutative) ring. Prove that the following conditions are equivalent:
   
   (a) \( R \) is semisimple.
   (b) Every left \( R \)-module is projective.
   (c) Every left \( R \)-module is injective.