1. Let $\rho : G \to \text{GL}_n(\mathbb{C})$ be a complex representation of a finite group $G$ and $\chi$ its associated character. Prove that $|\chi(g)| = \chi(1)$ if and only if $\rho(g) = \lambda I$ for some root of unity $\lambda$.

2. Let $G \neq 1$ be a finite group. Prove that $G$ is simple if and only if for all irreducible complex characters $\chi$ of $G$ and for all $g \neq 1$ one has $\chi(g) \neq \chi(1)$.

3. Let $G$ be a finite group and $G'$ its commutator subgroup. Prove that the number of complex linear (i.e., degree 1) characters of $G$ is $[G : G']$.

4. Let $k$ be a field of characteristic $p > 0$ and $R$ a finite-dimensional $k$-algebra. Let $M$ be a simple (left) $R$-module such that $p$ does not divide $\dim_k M$. Prove that $\chi_M \neq 0$.

5. Let $\rho : G \to \text{GL}_k(V)$ be a $k$-linear representation of a finite group $G$.
   
   (a) Prove that if $\rho(G)$ spans $\text{End}_k(V)$ as a $k$-vector space then $\rho$ is irreducible.
   
   (b) Prove that the converse to part (a) holds if $k$ is algebraically closed and $V$ is finite dimensional.
   
   (c) Let $G = C_4$ and $F = \mathbb{R}$. Show that $G$ has an irreducible $\mathbb{R}$-representation $\rho$ of degree 2 and the span of $\rho(G)$ is a proper subspace of $M_2(\mathbb{R})$.

6. Find the character table (over $\mathbb{C}$) for $D_8$, the dihedral group of order 8.

7. Let $G$ be a finite group.

   (a) For $g, h \in G$ prove that $g$ and $h$ are in the same conjugacy class of $G$ if and only if $\chi(g) = \chi(h)$ for every irreducible complex character $\chi$ of $G$.

   (b) Prove that $g$ is conjugate to $g^{-1}$ if and only if $\chi(g) \in \mathbb{R}$ for every irreducible complex character $\chi$ of $G$.

8. Let $\rho$ be a $\mathbb{C}$-linear representation of a finite group $G$ and $\chi$ its associated character. Suppose

   $$\frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = 3.$$

   Prove that $\rho$ is the direct sum of three distinct irreducible representations.