Let $G$ be a finite group and $\mathbb{C}$ the field of complex numbers. Then

$$\mathbb{C}[G] = B(I_1) \times \cdots \times B(I_t) \cong n_1 I_1 \oplus \cdots \oplus n_t I_t$$

where $I_1, \ldots, I_t$ represent all the simple left ideals of $\mathbb{C}[G]$ up to isomorphism and $B(I_j)$ is the sum of all the left ideals of $\mathbb{C}[G]$ which are isomorphic to $I_j$.

Then the following hold:

- $t$ equals the number of conjugacy classes of $G$.
- $n_i = \dim_{\mathbb{C}} I_i$ for $1 \leq i \leq t$.
- $B(I_i) \cong M_{n_i}(\mathbb{C})$ for $1 \leq i \leq t$.

Now let $C_1, \ldots, C_t$ be the conjugacy classes of $G$. Let $m_i = |C_i|$ for $i = 1, \ldots, t$. For each $i$, set $z_i = \sum_{g \in C_i} g \in \mathbb{C}[G]$. Let $e_i$ be the identity element of $B(I_i)$. Then

$$Z(\mathbb{C}[G]) = \mathbb{C}e_1 \times \cdots \times \mathbb{C}e_t = \mathbb{C}z_1 \oplus \cdots \oplus \mathbb{C}z_t.$$

For each $i$, let $\chi_i$ be the character associated to the $\mathbb{C}[G]$-module $I_i$. Then $\chi_1, \ldots, \chi_t$ are the distinct irreducible characters of $G$ (over $\mathbb{C}$). Recall that for any character $\chi$ of $G$, $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.

The elements $e_1, \ldots, e_t$ and $z_1, \ldots, z_t$ are related by the following theorem:

$$e_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g^{-1})g.$$ \hfill (1)

$$z_i = m_i \sum_{j=1}^t \frac{\chi_j(g)}{n_j} e_j \quad \text{for any } g \in C_i.$$ \hfill (2)

For class functions $\chi$ and $\phi$ of $G$ we define the (Hermitian) inner product

$$\langle \chi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\phi(g)}.$$

Then $\{\chi_1, \ldots, \chi_t\}$ form an orthonormal basis for the space of class functions on $G$. That is,

$$\sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij}|G|. \hfill (3)$$

Finally, for $g \in C_i$, $h \in C_j$, we have

$$\sum_{\ell=1}^t \chi_\ell(g)\overline{\chi_\ell(h)} = \delta_{ij} \frac{|G|}{m_i}. \hfill (4)$$