1. Prove that $R$ is a division ring if and only if $R$ has only two left ideals.

**Solution:** Suppose $R$ is a division ring and $I$ is a nonzero left ideal. Let $x \in I$, $x \neq 0$. Then $r = (rx^{-1})x \in I$ for every $r \in R$. Hence, $R = I$. Thus, $R$ has exactly two left ideals.

Suppose now that $R$ has only two left ideals (namely, $(0)$ and $R$). Let $x \in R$, $x \neq 0$. Then $Rx = R$, so there exists $y \in R$ such that $yx = 1$. Similarly, $Ry = R$, so there exists $z \in R$ such that $zy = 1$. Then $x = (zy)x = z(yx) = z$. Thus, $xy = 1 = yx$ and $x$ is invertible. Consequently, $R$ is a division ring.

2. Prove that a left Artinian domain is a division ring.

**Solution:** Let $x \in R$, $x \neq 0$. Consider the descending chain of left ideals $Rx \supseteq Rx^2 \supseteq Rx^3 \supseteq \cdots$. As $R$ is left Artinian, we have $Rx^n = Rx^{n+1}$ for some $n$. Hence, $x^n = rx^{n+1}$ for some $r \in R$; equivalently, $(1 - rx)x^n = 0$. As $R$ is a domain, we can cancel $x^n$ and obtain $1 = rx$; i.e., $x$ is left invertible. Since $x$ was an arbitrary nonzero element, this implies that $R$ has only two left ideals. By Problem #1, $R$ is a division ring.

3. A ring $R$ is called Dedekind-finite if for all $a, b \in R$, $ab = 1$ implies $ba = 1$.

(a) Prove that any domain is Dedekind-finite.

**Solution:** Suppose $ab = 1$. Then $(ba - 1)b = 0$. Since $b \neq 0$, this implies that $ba - 1 = 0$, so $ba = 1$.

(b) Prove that if $R$ is left Noetherian then $R$ is Dedekind-finite. (Hint: It might help to first prove that any surjective endomorphism of a left Noetherian module is an isomorphism.)

**Solution:** We first prove that statement in the Hint. Let $\phi : M \to M$ be a surjective homomorphism, where $M$ is a Noetherian left $R$-module. Consider the ascending chain of left submodules of $M$: $\ker \phi \subseteq \ker \phi^2 \subseteq \ker \phi^3 \subseteq \cdots$. Then there exists an $n$ such that $\ker \phi^n = \ker \phi^{n+i}$ for all $i \geq 0$. In particular, $\ker \phi^n = \ker \phi^{2n}$. Note that $\phi^n$ is a surjective homomorphism and that if we prove $\phi^n$ is injective, so is $\phi$. Thus, by replacing $\phi^n$ with $\phi$, we may assume without loss of generality that $\ker \phi = \ker \phi^2$. Let $x \in \ker \phi$. As $\phi$ is surjective, $x = \phi(y)$ for some $y \in M$. Then $\phi^2(y) = \phi(x) = 0$, so $y \in \ker \phi^2 = \ker \phi$. Hence, $x = \phi(y) = 0$. Thus, $\ker \phi = \{0\}$ and $\phi$ is injective.

Now, suppose $R$ is left Noetherian and $ab = 1$ for some $a, b \in R$. Consider the left $R$-module homomorphism $\phi : R \to R$ given by $\phi(r) = rb$. Since $\phi(ra) = rab = r$ for every $r \in R$, we see that $\phi$ is surjective. Hence, by the Hint, $\phi$ is an injective. As $\phi(ba) = bab = b = \phi(1)$, we conclude that $ba = 1$. Hence, $R$ is Dedekind-finite.
4. Let $S = \begin{pmatrix} Q & Q \\ 0 & \mathbb{Z} \end{pmatrix}$. Prove that $S$ is left Noetherian but not right Noetherian. (Do not quote the theorem from class, but rather prove this “from scratch”.)

**Solution:** We first show that $S$ is left Noetherian. Let $I$ be a nonzero left ideal of $S$. Assume that $I$ contains a matrix of the form $\begin{pmatrix} q_1 & q_2 \\ 0 & a \end{pmatrix}$ with $a \neq 0$. Then multiplying this matrix on the left by $\begin{pmatrix} 0 & a^{-1}q \\ 0 & 0 \end{pmatrix}$ we obtain $\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \in I$ for any $q \in Q$. Thus $I$ contains the (two-sided) ideal $J$ of $S$ consisting of all matrices in $S$ with zeros everywhere except the $(1,2)$-entry. Now consider the ring homomorphism $\phi : R \to Q \times \mathbb{Z}$ which sends $\begin{pmatrix} q_1 & q_2 \\ 0 & a \end{pmatrix}$ to $(q_1, a)$. Clearly, $\phi$ is surjective and $\ker \phi = J$. Then $R/J \cong Q \times \mathbb{Z}$, which is (left and right) Noetherian. Thus, the left ideal $I/J$ is finitely generated. Since $J = R \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is finitely generated, we obtain that $I$ is finitely generated.

Now consider the case where all $(2,2)$-entries of elements of $I$ are zero. Then $I$ is a $Q$-vector space given by

$$ q \cdot \begin{pmatrix} q_1 & q_2 \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} qq_1 & qq_2 \\ 0 & 0 \end{pmatrix}. $$

Note that for $A = \begin{pmatrix} q_1 & q_2 \\ 0 & a \end{pmatrix} \in S$ and $B \in I$, $AB = q_1B$. Hence, $SB = QB$. As $I$ is a $Q$-subspace of $Q^2$, it has a finite basis. Then $I$ is generated as a left $R$-module by this basis. Hence, $I$ is finitely generated as a left $R$-module and $R$ is left Noetherian.

To show $R$ is not right Noetherian, consider for each $n \geq 0$ the right ideal

$$ I_n = \left\{ \begin{pmatrix} 0 & b \\ 0 & \frac{1}{2^n} \end{pmatrix} \mid b \in \mathbb{Z} \right\}. $$

Note that $I_n \subsetneq I_{n+1}$ for all $n$, as $\begin{pmatrix} 0 & \frac{1}{2^{n+1}} \\ 0 & 0 \end{pmatrix}$ is in $I_{n+1}$ but not $I_n$. Hence, $R$ does not satisfy ACC on right ideals.

5. Let $R$ be a ring and $S = M_n(R)$. Prove there exists a bijection between the set of ideals of $R$ and the set of ideals of $S$ given by $I \mapsto M_n(I)$. Conclude that $R$ is simple if and only if $S$ is. (Hint: Let $J$ be an ideal of $S$ and let $I$ be the set consisting of all the $(1,1)$-entries of matrices in $J$. Show that $I$ is an ideal of $R$ and that $J = M_n(I)$.)

**Solution:** Let $I$ be an ideal of $R$. Then it is evident that $M_n(I)$ is an ideal of $S$. Suppose now that $J$ is an ideal of $S$. Let $I$ be the set of elements consisting of all the $(1,1)$ entries of matrices in $J$. It should be obvious that $0 \in I$ (as the zero matrix is in $J$) and that $I$ is closed under addition. Let $E_{ij}$ be the element of $S$ which has $1$ in the $i$th row and $j$th column and zeros elsewhere. Let $r \in R$ and $a \in I$. Let $A$ be the matrix in $J$ with $a$ in the $(1,1)$-entry. Then $rE_{11}A \in J$ and $A(rE_{11}) \in J$ as $J$ is an ideal of $S$. Also, $rE_{11}A$ has $ra$ as its $(1,1)$ entry, and $A(rE_{11})$ has $ar$ as its $(1,1)$-entry. Thus, $ra, ar \in I$ and $I$ is an ideal. Now suppose $B = [b_{ij}] \in J$. Then $E_{11}BE_{j1} \in J$ and has $b_{ij}$ as its $(1,1)$ entry. Thus, $b_{ij} \in I$ for all $i, j$ and so $B \in M_n(I)$. Now let
$C = [c_{ij}] \in M_n(I)$. For each $i, j$ let $D_{ij}$ be the matrix in $J$ with $c_{ij}$ as its $(1,1)$-entry. Then $F_{ij} = E_{i1}D_{ij}E_{1j} \in J$ and has $c_{ij}$ as its $(i,j)$-entry and zeros elsewhere. Then $C = \sum_{ij} F_{ij} \in J$. Consequently, $J = M_n(I)$. 