Math 901
Solutions to Homework # 2

1. Let $E$ and $L$ be subfields of a field $K$, and $F$ a subfield of both $E$ and $L$. Suppose $E/F$ and $L/F$ are normal. Prove that $EL/F$ is normal.

**Solution:** Let $\sigma : EL \to \overline{F}$ be a field embedding fixing $F$, where $\overline{F}$ is an algebraic closure of $F$. Then $\sigma|_E : E \to \overline{F}$ and $\sigma|_L : L \to \overline{F}$ are embeddings fixing $F$. Since $E/F$ and $L/F$ are normal, we have $\sigma(E) = E$ and $\sigma(L) = L$. Hence, $\sigma(EL) \subseteq EL$ (using, for example, problem 3 from Homework set 1). Thus, $EL/F$ is normal.

2. Let $E/F$ be a separable algebraic extension and $L$ the normal closure of $E/F$. Prove that $L/F$ is separable.

**Solution:** The normal closure $L$ of $E/F$ is the splitting field of the set of all polynomials $\text{Min}(\alpha, F)$ where $\alpha \in E$. Since $E/F$ is separable, each $\text{Min}(\alpha, F)$ is a separable polynomial. Thus, the roots of each $\text{Min}(\alpha, F)$ are separable. Since $L$ is $F$ adjoin a set of separable elements (i.e., the roots of $\text{Min}(\alpha, F)$ for each $\alpha \in E$), we see that $L/F$ is separable.

3. Let $E/F$ be normal field extension and $K = F^{\text{sep}}$ and $L = F^{\text{insep}}$ be the separable and purely inseparable closures, respectively, of $F$ in $E$. Prove that $E/K$ is purely inseparable, $E/L$ is separable, and $E = KL$.

**Solution:** If $\text{char } F = 0$ then $E = K$ and $L = F$ and the result is trivial. So assume $\text{char } F = p > 0$. Let $\alpha \in E$. Then $\alpha^{p^n}$ is separable over $F$ for some $n$. Hence, $\alpha^{p^n} \in K$. Thus, $E/K$ is purely inseparable. As $E/F$ is normal, we have $E/L$ is normal. Suppose $E/L$ is inseparable. Then, by a theorem proved in class, there exists $\alpha \in E \setminus L$ which is purely inseparable over $L$; i.e., $\beta = \alpha^{p^n} \in L$ for some $n$. But as $L/F$ is purely inseparable, there exists $m$ such that $\beta^{p^m} \in F$; i.e., $\alpha^{p^{n+m}} \in F$. Thus, $\alpha$ is purely inseparable over $F$, so $\alpha \in L$, a contradiction. Hence, $E/L$ is separable. As $E \supseteq KL \supseteq L$ and $E/L$ is separable, we have that $E/KL$ is separable. As $E \supseteq KL \supseteq K$ and $E/K$ is purely inseparable, we have that $E/KL$ is purely inseparable. Since the only elements which are both separable over $KL$ and purely inseparable over $KL$ are the elements of $KL$, we conclude that $E = KL$.

4. Let $E/F$ be a field extension. Prove that there exists a unique intermediate field $L$ of $E/F$ such that $E/L$ is purely inseparable and $L/F$ is separable.

**Solution:** We’ve seen in the previous problem that $L = F^{\text{sep}}$ satisfies the requisite properties. Suppose $T$ is an intermediate field such that $E/T$ is purely inseparable and $T/F$ is separable. Then, as $T/F$ is separable, $T \subseteq L$. On the other hand, $L/T$ is purely inseparable and also separable. Hence, $L = T$. 


5. Let $E/F$ be a normal field extension and $f(x) \in F[x]$ an irreducible polynomial. Suppose $g(x)$ and $h(x)$ are monic irreducible factors of $f(x)$ in $E[x]$. Prove that there exists an automorphism $\sigma$ of $E$ such that $g(x) = h^\sigma(x)$, where $h^\sigma(x)$ is the polynomial obtained by applying $\sigma$ to the coefficients of $h(x)$.

**Solution:** Let $\alpha, \beta \in \overline{E}$ be roots of $g(x)$ and $h(x)$, respectively. Then, as $\alpha$ and $\beta$ are both roots of $f(x)$ and $f(x)$ is irreducible over $F$, there exists an isomorphism $\tau : F(\alpha) \rightarrow F(\beta)$ fixing $F$ and such that $\tau(\alpha) = \beta$. Extend $\tau$ to $\phi : E \rightarrow E$. Let $\sigma : E \rightarrow E$ be the restriction of $\phi$ to $E$, which is an automorphism of $E$. (Here we are using $E/F$ is normal.) Note that $g(x) = \text{Min}(\alpha, E)$ and $h(x) = \text{Min}(\beta, E)$. Now,

$$0 = \phi(0) = \phi(g(\alpha)) = g^\phi(\phi(\alpha)) = g^\sigma(\beta).$$

As $g^\sigma(x)$ is monic, irreducible over $E$ and $\beta$ is a root, we must have $g^\sigma(x) = h(x)$.

6. Let $E = \mathbb{F}_p(t)$ where $t$ is transcendental over $\mathbb{F}_p$. Let $\sigma$ be the automorphism of $E$ which (necessarily) fixes $\mathbb{F}_p$ and such that $\sigma(t) = t + 1$. Let $E_\sigma = \{ \alpha \in E \mid \sigma(\alpha) = \alpha \}$. Prove that $E_\sigma = \mathbb{F}_p(t^p - t)$.

**Solution:** First note that $\sigma(t^p - t) = \sigma(t)^p - \sigma(t) = (t+1)^p - (t+1) = t^p + 1 - (t+1) = t^p - t$. Hence, $\mathbb{F}_p(t^p - t) \subseteq E_\sigma$. Also note that $[E : \mathbb{F}_p(t^p - t)] \leq p$ since $E = \mathbb{F}_p(t)$ and $t$ is a root of the polynomial $x^p - x - (t^p - t) \in \mathbb{F}_p(t^p - t)[x]$. On the other hand, by Artin’s Theorem, $[E : E_\sigma] = |\langle \sigma \rangle| = p$. (One easily checks that $\sigma^p(t) = t$.) Thus, $[E_\sigma : \mathbb{F}_p(t^p - t)] = 1$ and so $E_\sigma = \mathbb{F}_p(t^p - t)$. 