Math 901  
**Homework # 2**  
*Due: Friday, September 18th*

1. Let $E$ and $L$ be subfields of a field $K$, and $F$ a subfield of both $E$ and $L$. Suppose $E/F$ and $L/F$ are normal. Prove that $EL/F$ is normal.

2. Let $E/F$ be a separable algebraic extension and $L$ the normal closure of $E/F$. Prove that $L/F$ is separable.

3. Let $E/F$ be normal field extension and $K = F^{\text{sep}}$ and $L = F^{\text{insep}}$ be the separable and purely inseparable closures, respectively, of $F$ in $E$. Prove that $E/K$ is purely inseparable, $E/L$ is separable, and $E = KL$.

4. Let $E/F$ be a field extension. Prove that there exists a unique intermediate field $L$ of $E/F$ such that $E/L$ is purely inseparable and $L/F$ is separable.

5. Let $E/F$ be a normal field extension and $f(x) \in F[x]$ an irreducible polynomial. Suppose $g(x)$ and $h(x)$ are monic irreducible factors of $f(x)$ in $E[x]$. Prove that there exists an automorphism $\sigma$ of $E$ such that $g(x) = h^\sigma(x)$, where $h^\sigma(x)$ is the polynomial obtained by applying $\sigma$ to the coefficients of $h(x)$.

6. Let $E = \mathbb{F}_p(t)$ where $t$ is transcendental over $\mathbb{F}_p$. Let $\sigma$ be the automorphism of $E$ which (necessarily) fixes $\mathbb{F}_p$ and such that $\sigma(t) = t + 1$. Let $E_\sigma = \{ \alpha \in E \mid \sigma(\alpha) = \alpha \}$. Prove that $E_\sigma = \mathbb{F}_p(t^p - t)$. 