4. Let $E/F$ and $F/K$ be separable (algebraic) extensions (but not necessarily finite). Prove that $E/K$ is separable.

Solution: We first prove this in the case $E/K$ is a finite extension. As $E/F$ and $F/K$ are separable, we have that $[E : F] = [E : F]_s$ and $[F : K] = [F : K]_s$. Then $[E : K] = [E : F][F : K] = [E : F]_s[F : K]_s = [E : K]_s$. Hence, $E/K$ is separable.

Now consider the general case and let $\alpha \in E$ and $f(x) = \text{Min}(\alpha, F)$. Write $f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_0$, which is a polynomial in $F[x]$. Let $L = K(c_0, \ldots, c_{n-1})$. Then $L/K$ is a finite separable extension (separable since $F/K$ is separable). Since $L \subseteq F$ and $f(x)$ is irreducible in $F[x]$, $f(x)$ is irreducible in $L[x]$. Thus, $f(x) = \text{Min}(\alpha, L)$.

As $\alpha$ is separable over $F$, $f(x)$ has no multiple roots in its splitting field. Thus, $\alpha$ is separable over $L$ as well. Hence, $L(\alpha)/L$ is a separable (and finite) extension. By the finite case, we see that $L(\alpha)/K$ is a separable extension, and hence $\alpha$ is separable over $K$.

3. Let $K \subseteq E, F \subseteq L$ be fields. Suppose $E/K$ and $F/K$ are algebraic. Prove that $EF/K$ is algebraic and that

$$EF = \{e_1f_1 + \cdots e_nf_n \mid e_i \in E, f_i \in F, n \geq 1 \text{ arbitrary} \}.$$ 

Solution: Let $S$ be the algebraic closure of $K$ in $L$. By results shown in class, $S$ is a subfield of $L$. As $E/K$ and $F/K$ are algebraic, $E \cup F \subseteq S$. By definition of $EF$, $EF \subseteq S$. Hence, $EF$ is algebraic over $K$. To prove the second statement, let $T$ the set on the right side of the equality. Clearly, $T \subseteq EF$. Just as clearly, $T$ is closed under addition, subtraction, multiplication and contains both $E$ and $F$. So $T$ is a ring containing $K$. It suffices to show that $T$ is a field (as then $T \supseteq EF$). Suppose $\alpha \in T$, $\alpha \neq 0$. Then $K[\alpha] \subseteq T$, as $K[\alpha]$ is the smallest ring containing both $K$ and $\alpha$. But as $\alpha$ is algebraic over $K$ (as $EF$ is algebraic over $K$), $K[\alpha] = K(\alpha)$ is a field. So $\alpha^{-1} \in K[\alpha] \subseteq T$. Hence, $T$ is a field and $T = EF$.

4. Let $K \subseteq E, F \subseteq L$ be fields. Suppose $E/K$ and $F/K$ are separable (algebraic). Prove that $EF/K$ is separable and that $[EF : K]_s \leq [E : K]_s[F : K]_s$.

Solution: Let $S$ be the separable closure of $K$ in $L$. By results shown in class, $S$ is a field. As $E/K$ and $F/K$ are separable over $K$, we get that $E \cup F \subseteq S$. By definition of $EF$, $EF \subseteq S$. Hence, $EF$ is separable over $K$. For the second statement, we assume $E/K$ and $F/K$ are finite extensions. Also, as the extensions are all separable, it suffices to prove that $[EF : K] \leq [E : K][F : K]$. (Note: the inequality here holds whether or not the extensions are separable. You should try to prove it in this generality.) Certainly, $[EF : K] = [EF : F][F : K]$. Hence, it suffices to show that $[EF : F] \leq [E : K]$. Let $\beta_1, \ldots, \beta_n$ be a $K$-basis for $F$. So $F = K\beta_1 + \cdots + K\beta_n$. Then using the second statement of problem #3, we see that $EF = E\beta_1 + \cdots + E\beta_n$. (The coefficients from $K$ can be absorbed into the coefficients from $E$.) Hence, $[EF : F] \leq n = [F : K]$. 

Math 901

Solutions to selected problems from Homework #1
5. Let $F$ be a field of characteristic $p > 0$ and $a \in F$. Prove that $x^p - a \in F[x]$ is either irreducible or splits completely in $F[x]$.

**Solution:** Suppose $x^p - a$ is reducible. Let $\beta$ be a root of $x^p - a$ is some algebraic closure of $F$. Then $\beta^p = a$ and in $F[x]$, $x^p - a = x^p - \beta^p = (x - \beta)^p$. Let $f(x) = \text{Min}(\beta, F)$. Then $f(x)$ divides $(x - \beta)^p$, say $f(x) = (x - \beta)^i$, where $1 \leq i < p$. (If $i = p$, then $x^p - a$ is irreducible, contradicting our assumption.) Then $f(x) = x^i - i\beta x^{i-1} + \cdots + (-1)^i \beta^i$, so $i\beta \in F$. As $i$ is nonzero and hence a unit in $F$, we see that $\beta \in F$. Thus, $x^p - a = (x - \beta)^p$ splits completely in $F[x]$. 