Math 901
Exam II
Due: 11:30 am on Monday, December 7th

Instructions: Do at most 7 problems. All problems are worth 15 points. You must work alone, but you may use your notes and any other non-human resources (books, websites, etc) you wish. However, the problems can all be solved using just results we have discussed in class or in the homework, so your solutions should be confined to that material. (In other words, don’t use theorems we haven’t proved in class.)

Throughout $R$ denotes a ring with identity.

1. Let $F$ be an algebraically closed field, $c \in F \setminus \{0\}$, and $n$ a positive integer. Prove that $F[x]/(x^n - c)$ is semisimple if and only if $\text{Char } F$ does not divide $n$.

2. Let $R = \prod_a R_a$ be a product of (possibly infinitely many) rings. Prove that $J(R) = \prod_a J(R_a)$, where $J(\cdot)$ denotes the Jacobson radical.

3. Let $R$ be a left Artinian ring. Prove that $R$ has only finitely many (two-sided) ideals containing its Jacobson radical.

4. Let $R$ be a left Artinian ring and $M, N$ simple $R$-modules such that $\text{Ann}_R M = \text{Ann}_R N$. Prove that $M \cong N$.

5. Let $R$ be a left Artinian ring and $K$ the sum of all simple left ideals of $R$. Prove that $K$ is an ideal of $R$ and that every nonzero left ideal $I$ intesects $K$ nontrivially, i.e, $I \cap K \neq 0$.

6. Let $D$ be a division ring and $V$ a $D$-vector space. Prove that $\text{End}_D(V)$ is a simple ring if and only if $\dim_D V < \infty$.

7. Let $P$ be a projective $R$-module and $a \in R$ which is not a zero-divisor on $R$. Suppose $au = 0$ for some $u \in P$. Prove that $u = 0$.

8. Let $R$ be a semisimple ring and $E = R^n$ a finitely generated free module. For $r \in R$, let $\ell_r$ denote the map from $E$ to $E$ given by left multiplication by $r$. Prove that $Z(\text{End}_R(E)) = \{\ell_r \mid r \in Z(R)\}$. (Here $Z(\cdot)$ denotes the center of a ring.)