**Math 818**

**Final Exam**

May 5, 2008

**Instructions:** Do one problem from each of the first three sections, and two problems from the fourth section. All problems are worth 20 points.
**Note:** All rings on this exam are assumed to be commutative with identity.

## I. Module Theory

**1.** Let $R$ be a ring and $M$ an $R$-module and $N$ a submodule of $M$. Prove that $M$ is Noetherian if and only if $N$ and $M/N$ are Noetherian.

**2.** Let $R$ be a domain and $M$ a finitely generated torsion $R$-module. Prove there exists a nonzero element $r \in R$ such that $rM = 0$, i.e., $r$ annihilates $M$. Give an example of a torsion $\mathbb{Z}$-module $M$ such that no nonzero element of $\mathbb{Z}$ annihilates $M$.

## II. Linear Algebra

**3.** Consider the matrix $A$ below with rational coefficients. (Note that this matrix is in a 'special' form.)

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) Find the rank of $A$.

(b) Find the invariant factors and the characteristic polynomial of $A$.

(c) Find the eigenvalues of $A$.

(d) Find the Jordan canonical form of $A$.

**4.** Let $F$ be a field and $V$ an $F$-vector space (but not necessarily of finite dimension). Let $S$ be a subset of $V$ which spans $V$. Prove that $S$ contains a basis for $V$.

## III. Field Theory

**5.** Let $E/F$ be a finite field extension. Suppose $f : E \to E$ is a field homomorphism which fixes $F$. Prove that $f$ is an automorphism of $E$.

**6.** Prove that the group of units of a finite field is cyclic.

## IV. Miscellaneous

**7.** Let $E$ be the splitting field for $x^5 - 2$ over $\mathbb{Q}$ and $G = \mathrm{Gal}(E/\mathbb{Q})$. Find $|G|$ and a set of generators for $G$. Also, find generators for a normal subgroup $H$ of $G$ such that $G/H$ is cyclic of order 4.

**8.** Let $F$ be a finite field. Prove that $F[x]$ has an irreducible polynomial of degree $m$ for every $m \geq 1$.

**9.** Let $R$ be a commutative ring and $M$ an $R$-module. Let $M^*$ denote the $R$-module $\mathrm{Hom}_R(M, R)$. For $m \in M$, let $\mathrm{ev}_m : M^* \to R$ be the map defined by $\mathrm{ev}_m(f) = f(m)$ for all $f \in M^*$.

(a) Prove $\mathrm{ev}_m \in \mathrm{Hom}_R(M^*, R) = M^{**}$.

(b) Prove the map $\phi : M \to M^{**}$ defined by $\phi(m) = \mathrm{ev}_m$ is a $R$-module homomorphism.

(c) Assume $R$ is a domain and $\phi : M \to M^{**}$ is injective. Prove that $M$ is torsion-free.

**10.** Let $F$ be a field and $A$ a square matrix with entries from $F$. Prove that $A$ is similar to $A^T$, the transpose of $A$.