Math 818

Exam # 2

Due: 11:30 am, April 21

Do four of the seven problems below. (All problems are worth 25 points.) If you wish, you can do a fifth problem for a possible additional 10 points, but this problem will be graded more stringently than the other four. (If you do this, be sure to identify that which problem is the "bonus" one.) You may use your notes and any textbooks you wish. However, the only person you may consult regarding this exam is me.

1. Let E be a splitting field of $x^6 + 5$ over $\mathbb{Q}$. Find $[E : \mathbb{Q}]$. Justify all details.

2. Let $f(x)$ be an irreducible polynomial of degree n over a field F. Let $g(x)$ be any non-constant polynomial in $F[x]$ and set $h(x) = f(g(x))$. Prove that if $q(x) \in F[x]$ is any irreducible factor of $h(x)$ then $\deg q$ is divisible by n. (Hint: If α is a root of $h(x)$ then $g(\alpha)$ is a root of $f(x)$.)

3. Let E be a splitting field for $x^4 - 4x^2 - 1$ over $\mathbb{Q}$. Find the Galois group of $E/\mathbb{Q}$. That is, explicitly describe all automorphisms of E and identify the group structure (e.g., by showing it is isomorphic to some well-known group).

4. Let E/F be a finite Galois field extension and K and L intermediate fields. Suppose $E = KL$ and $F = K \cap L$.
   (a) Give an example of fields E, F, K, L as above such that $[E : F] \neq [K : F][L : F]$.
   (b) If either K or L is normal over F, prove that $[E : F] = [K : F][L : F]$.

5. Let ω be a primitive 7th root of unity and let $E = \mathbb{Q}(\omega)$. Find all subfields of E and primitive elements (over $\mathbb{Q}$) for each.

6. Let E/F be a finite Galois extension and K an intermediate field. Let $G = \mathrm{Gal}(E/F)$ and $H = \mathrm{Gal}(E/K)$. Let $N_G(H)$ denote the normalizer of H in G. Prove that $N_G(H) = \{g \in G \mid g(K) = K\}$ and $N_G(H)/H \cong \mathrm{Aut}(K/F)$.

7. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and F a maximal subfield of $\overline{\mathbb{Q}}$ not containing $\sqrt{5}$ (such a field exists by Zorn's lemma). Let E be a finite normal extension of F. Prove that $\mathrm{Gal}(E/\mathbb{Q})$ is cyclic.