**Math 818**

**Exam I**

**Instructions:** Do a total of five problems, but no more than three from either section. (That is, do two from one section and three from the other.) All problems are worth 20 points.

**Note:** In all problems, $R$ denotes a commutative ring with identity.

**Section I**

1. Prove that any PID is a UFD. (You may use the fact that we proved in class that a domain $R$ is a UFD if and only if $R$ satisfies ACC on principal ideals and every irreducible element is prime.)

2. Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. (You may use that the function $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{N}$ defined by $N(a + b\sqrt{-5}) = a^2 + 5b^2$ satisfies $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}].$

3. Suppose $R$ is a Noetherian ring and $x$ is an indeterminate. Prove that $R[x]$ is Noetherian.

4. Let $R$ be a UFD and $P$ a prime ideal of $R$, and $F$ the field of fractions of $R$. Let $x$ be an indeterminate and $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in R[x]$. Suppose $a_n \notin P$, $a_i \in P$ for $0 \leq i \leq n-1$, and $a_0 \notin P^2$. Prove that $f(x)$ is irreducible in $F[x]$.

**Section II**

5. Let $R$ be a domain and $M$ an $R$-module. A subset $S$ of $M$ is called a maximal linearly independent set of $M$ if $S$ is linearly independent (over $R$) and any subset of $M$ properly containing $S$ is linearly dependent. Let $T$ be a linearly independent subset of $M$ and $N$ the $R$-submodule of $M$ generated by $T$. Prove that $T$ is a maximal linearly independent subset if and only if $M/N$ is torsion (i.e., Tor$(M/N) = M/N$).

6. Let $R$ be a ring and let $J$ be the intersection of all maximal ideals of $R$. For $x \in R$, prove that $x \in J$ if and only if $1 - rx$ is a unit for every $r \in R$.

7. Let $p$ be a positive prime integer and $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}[x]$. Let $I = (p, f(x))\mathbb{Z}[x]$. Find all maximal ideals of $\mathbb{Z}[x]/I$. (Hint: Note that $\overline{f}(x) \in \mathbb{Z}_p[x]$ is not irreducible unless $p = 2$.)

8. Let $R$ be a commutative ring.
   
   (a) Suppose $f : M \to N$ is a surjective homomorphism and $I$ an ideal of $R$. Prove that the function $\overline{f} : M/IM \to N/IN$ given by $\overline{f}(u + IM) = f(u) + IN$ is a well-defined surjective homomorphism.

   (b) Suppose $f : R^m \to R^n$ is a surjective homomorphism. Prove that $m \geq n$. 