Math 818

Exam 1

Instructions: Mix and Match: Do as many problems (or parts of problems) as you like as long as the total point value does not exceed 100.

Note: In all problems, R denotes a commutative ring with identity.

1. Let M be an R-module and $\phi : M \to M$ a surjective R-module homomorphism.

(a) (12 points) Suppose M is Noetherian. Prove that ϕ is an isomorphism.

(b) (6 points) Give an example of such an M and ϕ such that ϕ is not an isomorphism.

2. Let R be a domain and M an R-module. Recall that a subset S of M is called a *maximal linearly independent set* of M if S is linearly independent and any subset of M properly containing S is linearly dependent.

(a) (13 points) Let T be a linearly independent subset of M. Prove that T is contained in some maximal linearly independent subset of M.

(b) (7 points) Let T be a linearly independent subset of M and N the R-submodule of M generated by T. Prove that T is a maximal linearly independent subset if and only if M/N is torsion.

3. Let I be an ideal of R.

(a) (8 points) Prove that I is a free R-module if and only if $I = (0)$ or $I = (a)$ for some $a \in R$ which is not a zero-divisor.

(b) (5 points) Give an example of a submodule of a free R-module which is not free.

4. Let I be an ideal of R and M a finitely generated R-module.

(a) (15 points) Suppose $IM = M$. Prove there exists an element $a \in I$ such that $(1 - a)M = 0$. (Hint: Use the"determinant trick".)

(b) (7 points) Suppose I is finitely generated and $I^2 = I$. Prove that $I = (e)$ for some $e \in R$ such that $e^2 = e$. (Use part (a) with $M = I$.)

5. (10 points) Let F be a field and V a finite dimensional F-vector space. Let W be a subspace of V. Prove that $\dim V = \dim W + \dim V/W$.

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix over a field F. Define the *trace* of A, denoted $\operatorname{tr}(A)$, by $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$.

(a) (8 points) Prove that for $n \times n$ matrices A, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(b) (3 points) If A and B are similar, prove that $\operatorname{tr}(A) = \operatorname{tr}(B)$. (Use (a).)

(c) (10 points) Let A be a 2×2 matrix which is not a scalar matrix (i.e., not a scalar multiple of the identity matrix). Prove that A is similar to a unique matrix of the form:

$$\begin{bmatrix} 0 & -\det(A) \\ 1 & \operatorname{tr}(A) \end{bmatrix}.$$

7. Let A be a matrix with rational entries whose characteristic polynomial is $c(x) = x^2(x-1)^2(x+1)$.

(a) (10 points) Find all possible sets of invariant factors for A.

(b) (4 points) Find a specific matrix A as above whose minimal polynomial has degree 3.

8. (13 points) Find the rational canonical form of the following matrix in $M_3(\mathbb{Q})$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$