[1] (20 points) Let $R$ be a ring.
   (a) Give the definition of what it means for $R$ to be a *domain*. (Note: You do not 
       have to define “ring”, just give the conditions which say when a ring $R$ is a 
       domain.)

   (b) Give the definition of what it means for $R$ to be a *field*.

   (c) Prove that every field is a domain.

   (d) Give an example of a domain which is not a field.
[2] (25 points) Let \( n \geq 2 \) be an integer and let \([a]_n\) be an element of \( \mathbb{Z}_n \).

(a) Suppose \( \gcd(a, n) = 1 \). Prove that \([a]_n\) is a unit.

(b) Suppose \( \gcd(a, n) \neq 1 \). Prove that \([a]_n\) is a zero-divisor.

(c) Prove that \( \mathbb{Z}_n \) is a field if and only if \( n \) is prime. (You may use the results of (a) and (b) above even if you did not prove them.)
[3] (10 points) Let $R$ be a ring and suppose $a$ is a unit of $R$. Prove that the multiplicative inverse of $a$ is unique (i.e., that there is only one multiplicative inverse of $a$).

[4] (15 points) Give examples of each of the following.

(a) A noncommutative ring.

(b) A ring with exactly 12 units.

(c) A nonconstant polynomial in $\mathbb{Z}_{12}[x]$ which is a unit.

[5] (10 points) Let $f(x) = x^5 + 2x^3 - x^2 + 2x + 2$ and $g(x) = 2x^2 + x$ be polynomials in $\mathbb{Z}_3[x]$. Find $q(x), r(x) \in \mathbb{Z}_3[x]$ with $\deg r < \deg g$ such that $f(x) = g(x)q(x) + r(x)$. 
[6] (10 points) Let \( \epsilon(x) = \ln(x^{17}, 77) \) be the encryption function for an RSA cryptosystem. Find the decryption function.

[7] (10 points) Use Euler’s Theorem to find \( \lnr(13^{121}, 75) \).

[8] (10 points) **Bonus problem:** Let \( m = 2^n \) where \( n \geq 1 \) and let \( a \) be an integer. Prove that if \( a \) is odd then \( a^m \equiv 1 \pmod{m} \), and if \( a \) is even then \( a^m \equiv 0 \pmod{m} \).