1. Mathematical Induction

Let $n$ denote an arbitrary integer and $P(n)$ some mathematical sentence involving $n$. Here, $P(n)$ could be a formula or an inequality involving $n$, or some other type of sentence such as “$2n$ is the sum of two primes.”

Mathematical induction provides a way of establishing the validity of a sentence $P(n)$ for an infinite set of integers of the form $\{n \in \mathbb{Z} \mid n \geq r\}$ for some $r \in \mathbb{Z}$. There are two versions:

**Principle of Mathematical Induction (PMI):** Let $P(n)$ be a closed mathematical sentence and $r \in \mathbb{Z}$. Suppose the following:

1. $P(r)$ is true.
2. For every $n \geq r$, if $P(n)$ is true then $P(n+1)$ is true.

Then $P(n)$ is true for every $n \geq r$.

Here is a more powerful version:

**Principle of Complete Induction (PCI):** Let $P(n)$ be a closed mathematical sentence and $r \in \mathbb{Z}$. Suppose the following:

1. $P(r)$ is true.
2. For every $n > r$, if $P(k)$ is true for all $r \leq k < n$ then $P(n)$ is true.

Then $P(n)$ is true for every $n \geq r$.

Let’s try a couple of examples:

**Example:** Let’s prove that for all $n \geq 1$ that

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

**Solution:** For $n \geq 1$ let $P(n)$ denote the statement

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

We will use PMI to prove this holds for all $n \geq 1$. We first need to see that part (1) of PMI (called the “base case”) is true. But this is clear, since $\sum_{i=1}^{1} i = 1 = \frac{(1)(2)}{2}$. To establish the validity of part (2) of PMI (called the “induction step”), we suppose that $P(n)$ is true for some $n \geq 1$. We need to show $P(n+1)$ is true, which says that $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$. Note that

$$\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{(since $P(n)$ is true)}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2},$$
which is what we wanted to show. Since we have shown both parts of PMI hold for this \( P(n) \),
we conclude that \( P(n) \) is true for all \( n \geq 1 \).

Now let’s try one which needs PCI:

**Example:** The Fibonacci sequence is a sequence of integers \( \{a_n \mid n \geq 0\} \) defined recursively as follows: \( a_0 = a_1 = 1 \) and for \( n \geq 2 \), \( a_n = a_{n-1} + a_{n-2} \). Thus, \( a_2 = a_1 + a_0 = 2 \), \( a_3 = a_2 + a_1 = 3 \), \( a_4 = a_3 + a_2 = 5 \), etc. Let’s prove that \( a_n \leq \left( \frac{7}{4} \right)^n \) for all \( n \geq 0 \).

**Solution:** We first establish the base case: \( P(0) \) says that \( a_0 = 1 \leq \left( \frac{7}{4} \right)^0 = 1 \), which certainly is true. Since the recursion formula doesn’t kick in until \( n = 2 \), we need to establish \( P(1) \) separately. This is also clear, as \( a_1 = 1 \leq \left( \frac{7}{4} \right)^1 \). Now for the induction step: Suppose \( n > 1 \) and that \( P(k) \) is true for all integers \( k \) with \( 0 \leq k < n \). In particular, this means that \( P(n-1) \) and \( P(n-2) \) both hold; i.e., \( a_{n-1} \leq \left( \frac{7}{4} \right)^{n-1} \) and \( a_{n-2} \leq \left( \frac{7}{4} \right)^{n-2} \). Then

\[
a_n = a_{n-1} + a_{n-2} \\
\leq \left( \frac{7}{4} \right)^{n-1} + \left( \frac{7}{4} \right)^{n-2} \\
= \left( \frac{7}{4} \right)^{n-2} \left( \frac{7}{4} + 1 \right) \\
= \left( \frac{7}{4} \right)^{n-2} \left( \frac{11}{4} \right) \\
\leq \left( \frac{7}{4} \right)^{n-2} \left( \frac{7}{4} \right)^2 \quad \text{(since } \frac{11}{4} \leq \left( \frac{7}{4} \right)^2 \text{)} \\
= \left( \frac{7}{4} \right)^n.
\]

Thus, \( P(n) \) holds, which establishes the induction step. By PCI, \( P(n) \) is true for all \( n \geq 0 \).

Here are some for you to try:

**In Class Exercises:**

1. Let \( x \) be a positive real number. Prove by induction that \((1 + x)^n \geq 1 + nx\) for all integers \( n \geq 1 \).

2. Prove that for all integers \( n \geq 1 \)

\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.
\]

3. Prove that for all integers \( n \geq 1 \)

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.
\]
4. Define a sequence \( \{a_n\} \) as follows: \( a_0 = 1, \ a_1 = 2 \) and \( a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 2 \). Prove that \( a_n = 2^n \) for \( n \geq 0 \).

**Homework:**

1. Let \( n \) be a positive integer. Prove that
   \[
   \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.
   \]

2. Let \( \{a_n\} \) be the Fibonacci sequence. Prove that for all \( n \geq 0 \),
   \[
   a_n \leq \left(\frac{1 + \sqrt{5}}{2}\right)^n.
   \]

3. Prove that for all \( n \geq 1 \),
   \[
   \sum_{i=n}^{2n-1} (2i + 1) = 3n^2.
   \]