THE FROBENIUS FUNCTOR AND INJECTIVE MODULES

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Abstract. We investigate commutative Noetherian rings of prime characteristic such that the Frobenius functor applied to any injective module is again injective. We characterize the class of one-dimensional local rings with this property and show that it includes all one-dimensional F-pure rings. We also give a characterization of Gorenstein local rings in terms of $\operatorname{Tor}_i^R(R^f, E)$, where E is the injective hull of the residue field and R^f is the ring R whose right R-module action is given by the Frobenius map.

1. Introduction

Let R be a commutative Noetherian ring of prime characteristic p and $f : R \longrightarrow R$ the Frobenius ring homomorphism (i.e., $f(r) = r^p$ for $r \in R$). We let R^f denote the ring R with the $R - R$ bimodule structure given by $r \cdot s := rs$ and $s \cdot r := sf(r)$ for $r \in R$ and $s \in R^f$. Then $F_R(-) := R^f \otimes_R -$ is a right exact functor on the category of (left) R-modules and is called the *Frobenius functor* on R. This functor has played an essential role in the solution of many important problems in commutative algebra for local rings of prime characteristic (e.g., [9], [14], [15]). Of particular interest is how properties of the Frobenius map (or functor) characterize classical properties of the ring. The most important result of this type, proved by Kunz [11], says that F_R is exact if and only if R is a regular ring. As another example, Iyengar and Sather-Wagstaff prove that a local ring R is Gorenstein if and only if R^f (viewed as a right R-module) has finite G-dimension [12].

As F_R is additive and $F_R(R) \cong R$, it is easily seen that F_R preserves projective (in fact, flat) modules. In this paper, we consider rings for which F_R preserves injective modules, i.e., rings R having the property that $F_R(I)$ is injective for every injective R-module I. It is

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well known (e.g., [10]) that Gorenstein rings have this property, and in fact this is true for quasi-Gorenstein rings as well (Proposition 3.4). In Section 3, we show that if F_R preserves injectives then R satisfies Serre's condition S_1 and that $F_R(I) \cong I$ for every injective R-module I. Moreover, if R satisfies S_2 and is the homomorphic image of a Gorenstein local ring then F_R preserves all injectives if and only if $F_R(E)$ is injective, where E is the injective hull of the residue field. We also give a criterion (Theorem 3.8) for a local ring R to be Gorenstein in terms of $\operatorname{Tor}_i^R(R^f, E)$:

Theorem 1.1. *Let (R, m) be a local ring and $E = E_R(R/m)$. Then the following are equivalent:*

(a) R is Gorenstein;
(b) $\operatorname{Tor}_0^R(R^f, E) \cong E$ and $\operatorname{Tor}_i^R(R^f, E) = 0$ for $i = 1, \ldots, \operatorname{depth} R$.

In Section 4, we study one-dimensional rings R such that F_R preserves injectives. In particular, we give the following characterization (Theorem 4.1) in the case R is local:

Theorem 1.2. *Let (R, m) be a one-dimensional local ring and $E = E_R(R/m)$. The following conditions are equivalent:*

(a) $F_R(E)$ is injective;
(b) $F_R(I) \cong I$ for all injective R-modules I;
(c) R is Cohen-Macaulay and has a canonical ideal ω_R such that $\omega_R \cong \omega_R^{[p]}$.

Using this characterization, we show that every one-dimensional F-pure ring preserves injectives. We also prove, using a result of Goto [5], that if R is a complete one-dimensional local ring with algebraically closed residue field and has at most two associated primes then R is Gorenstein if and only if F_R preserves injectives.

In Section 2, we summarize several results concerning the Frobenius functor and canonical modules which will be needed in the later sections. Most of these are well-known, but for some we could not find a reference in the literature.

Finally, we remark that Theorems 1.1 and 1.2 are 'dual', in some sense, to results in [5] under the hypothesis that the Frobenius map is finite. That is, the functor $R^f \otimes_R -$ is replaced by $\operatorname{Hom}_R(R^f, -)$, the module E is replaced by R, etc. However, we do not see a method to directly deduce our results from those in [5] (or vice-versa), even in the F-finite case.

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2. Some properties of the Frobenius functor and canonical modules

Throughout this paper R denotes a commutative Noetherian ring of prime characteristic p. For an R-module M, $E_R(M)$ will denote the injective hull of M. If I is an ideal of R then $H^i_I(M)$ will denote the ith local cohomology module of M with support in I. If R is local with maximal ideal m, we denote the m-adic completion of R by $\widehat{R}$. We refer the reader to [2] or [13] for any unexplained terminology or notation.

Let M be a finitely generated R-module with presentation $R^r \xrightarrow{\varphi} R^s \longrightarrow M \longrightarrow 0$, where φ is represented (after fixing bases) by an $s \times r$ matrix A. Then $F_R(M)$ has the presentation $R^r \xrightarrow{F_R(\varphi)} R^s \longrightarrow F_R(M) \longrightarrow 0$ and the map $F_R(\varphi)$ is represented by the matrix $A^{[p]}$ obtained by raising the corresponding entries of A to the pth power. For an ideal I of R and $q = p^e$, we let $I^{[q]}$ denote the ideal generated by the set $\{i^q \mid i \in I\}$. Note that, by above, $F^e_R(R/I) \cong R/I^{[q]}$, where F^e_R is the functor F_R iterated e times.

The following proposition lists a few properties of the Frobenius functor which we will use in the sequel. Most of these are well known:

Proposition 2.1. *The following hold for any R-module M:*

(a) If T is an R-algebra then $F_T(T \otimes_R M) \cong T \otimes_R F_R(M)$.

(b) If S is a multiplicatively closed set of R then $F_R(M_S) \cong F_{R_S}(M_S) \cong R_S \otimes_R F_R(M)$.

(c) $\operatorname{Supp}_R F_R(M) = \operatorname{Supp}_R M$.

(d) M is Artinian if and only if $F_R(M)$ is Artinian.

(e) If (R, m) is local, and M is finitely generated of dimension s, then $F_R(H^s_m(M)) \cong H^s_m(F_R(M))$.

Proof. The first two parts follow easily from properties of tensor products. For (c), it suffices to show $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R F_R(M)$. Using (a) and (b), it is enough to prove that if R is a complete local ring and $M \neq 0$ then $F_R(M) \neq 0$. Then $R = S/I$ where S is a regular local ring of characteristic p. Let $Q \in \operatorname{Ass}_S M$. Then there is an exact sequence $0 \longrightarrow S/Q \longrightarrow M$. As S is regular, F_S is exact and we have an injection $S/Q^{[p]} \longrightarrow F_S(M)$. Hence, $F_S(M) \neq 0$. By part (b) $F_R(M) \cong S/I \otimes_S F_S(M)$. As $IM = 0$, $I^{[p]} F_S(M) = 0$. Let t be an integer such that $I^t \subseteq I^{[p]}$. Now, if $F_R(M) = 0$ then $F_S(M) = IF_S(M)$. Iterating, we have $F_S(M) = I^t F_S(M) = 0$, a contradiction. Hence, $F_R(M) \neq 0$.

For (d), since $\operatorname{Supp}_R M = \operatorname{Supp}_R F_R(M)$ and the support of an Artinian module is finite, it suffices to consider the case (R, m) is a local ring. Furthemore, for any R-module N with $\operatorname{Supp}_R N \subseteq \{m\}$, we have $\widehat{R} \otimes_R N \cong N$. Hence, we may assume R is complete. Then $R = S/I$ where S is a regular local ring of characteristic p. As $F_R(M) \cong S/I \otimes_S F_S(M)$, it is clear that if $F_S(M)$ is Artinian then so is $F_R(M)$. Conversely, if $S/I \otimes_S F_S(M)$ is Artinian, then $F_S(M)$ is Artinian since we also have $I^{[p]} F_S(M) = 0$. Thus, it is enough to prove the result in the case that R is a regular local ring. Recall that an R-module is Artinian if and only if $\operatorname{Supp}_R M \subseteq \{m\}$ and $(0 :_M m)$ is finitely generated. Since $\operatorname{Supp}_R M = \operatorname{Supp}_R F_R(M)$, it suffices to prove that $(0 :_M m)$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated. As F_R is exact, $F_R((0 :_M m)) \cong (0 :_{F_R(M)} m^{[p]})$. But $(0 :_M m)$ is finitely generated if and only if $F_R((0 :_M m))$ is finitely generated, and $(0 :_{F_R(M)} m^{[p]})$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated.

For (e), let $I = \operatorname{Ann}_R M$ and choose $x_1 \ldots, x_s \in m$ such that their images in R/I form a system of parameters. Set $J = (x_1, \ldots, x_s)$. Then $H^s_m(M) \cong H^s_J(M)$. Since $H^i_J(R) = 0$ for all $i > s$, $T \otimes_R H^s_J(M) \cong H^i_{TJ}(T \otimes_R M)$ for any R-algebra T. Then

$$F_R(H^s_m(M)) \cong R^f \otimes_R H^s_J(M) \cong H^s_{J^{[p]}}(F_R(M)).$$

Finally, as $I^{[p]} \subseteq \operatorname{Ann}_R F_R(M)$ and $J^{[p]} + I^{[p]}$ is m-primary, we have $H^s_{J^{[p]}}(F_R(M)) \cong H^s_m(F_R(M))$. □

We need one more result concerning the Frobenius, which is again well known:

Lemma 2.2. *Let (R, m) be a local ring of dimension d. If R is Cohen-Macaulay then $\operatorname{Tor}^R_i(R^f, H^d_m(R)) = 0$ for all $i \geq 1$.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_d$ be a system of parameters for R and $C(\mathbf{x})$ the Čech cochain complex with respect to $\mathbf{x}$. Note that $F_R(C(\mathbf{x})) \cong C(\mathbf{x}^p)$ where $\mathbf{x}^p = x_1^p, \ldots, x_d^p$. Since R

is Cohen-Macaulay, $\mathbf{x}$ is a regular sequence and thus $C(\mathbf{x})$ is a flat resolution of $H^d_m(R)$. Hence for $i \geq 1$,

$$\operatorname{Tor}_i^R(R^f, H^d_m(R)) \cong \mathrm{H}^{d-i}(R^f \otimes_R C(\mathbf{x})) \cong \mathrm{H}^{d-i}(C(\mathbf{x}^p)) = 0.$$

□

For a nonzero finitely generated R-module M we let $U_R(M)$ be the intersection of all the primary components Q of 0 in M such that $\dim M/Q = \dim M$. It is easily seen that $U_R(M) = \{x \in M \mid \dim Rx < \dim M\}$. A local ring R is said to be *unmixed* if $U_{\widehat{R}}(\widehat{R}) = 0$.

Let (R, m) be a local ring of dimension d, $E = E_R(R/m)$, and $(-)^{\vee} := \operatorname{Hom}_R(-, E)$ the Matlis dual functor. A finitely generated R-module K is called a *canonical module* of R if $K \otimes_R \widehat{R} \cong H^d_m(R)^{\vee}$. If a canonical module exists, it is unique up to isomorphism and denoted by ω_R. Any complete local ring possesses a canonical module. More generally, R possesses a canonical module if R is the homomorphic image of a Gorenstein ring. Proofs of these facts can be found in [1] (or the references cited there). We summarize some additional properties of canonical modules in the following proposition:

Proposition 2.3. *Let R be a local ring which possesses a canonical module ω_R and let $h : R \longrightarrow \operatorname{Hom}_R(\omega_R, \omega_R)$ be the natural map. The following hold:*

(a) $\operatorname{Ann}_R \omega_R = U_R(R)$.

(b) $(\omega_R)_P \cong \omega_{R_P}$ *for every prime* $P \in \operatorname{Supp}_R \omega_R$.

(c) $\omega_R \otimes_R \widehat{R} \cong \omega_{\widehat{R}}$.

(d) $\ker h = U_R(R)$.

(e) h is an isomorphism if and only if R satisfies Serre's condition S_2.

(f) If R is complete, $\operatorname{Hom}_R(M, \omega_R) \cong H^d_m(M)^{\vee}$ *for any R-module M.*

Proof. The proofs of parts (a)-(e) can be found in [1]. Part (f) is just local duality, but it can also be seen directly from the definition of ω_R and adjointness:

$$\operatorname{Hom}_R(H^d_m(M), E) \cong \operatorname{Hom}_R(M \otimes_R H^d_m(R), E) \cong \operatorname{Hom}_R(M, H^d_m(R)^{\vee}).$$

□

If R is a local ring possessing a canonical module ω_R such that $\omega_R \cong R$, then R is said to be *quasi-Gorenstein*. Equivalently, R is quasi-Gorenstein if and only if $H^d_m(R) \cong E$. By the proposition above, if R is quasi-Gorenstein then R is S_2 and R_P is quasi-Gorenstein for every $P \in \operatorname{Spec} R$. It is easily seen that R is Gorenstein if and only if R is Cohen-Macaulay and quasi-Gorenstein. Finally, there exist quasi-Gorenstein rings which are not Cohen-Macaulay (e.g., [1]).

3. Rings for which Frobenius preserves injectives

To facilitate our discussion we make the following definition:

Definition 3.1. A Noetherian ring of characteristic p is said to be *FPI* (i.e., 'Frobenius Preserves Injectives') if $F_R(I)$ is injective for every injective R-module I. We say that R is *weakly FPI* if $F_R(I)$ is injective for every Artinian injective R-module I.

We note that R is FPI (respectively, weakly FPI) if $F_R(E_R(R/P))$ is injective for every prime (respectively, maximal) ideal P of R. Also, since Frobenius commutes with localization, R is FPI if and only if R_P is weakly FPI for every prime ideal P.

Proposition 3.2. *Let I be an injective R-module and suppose $F_R(I)$ is injective. Then $F_R(I) \cong I$.*

Proof. By the remarks above, it suffices to prove this in the case R is local and $I = E_R(R/m)$ where m is the maximal ideal. And as $E_R(R/m) \cong E_{\widehat{R}}(\widehat{R}/\widehat{m})$, we may also assume R is complete. Let $E = E_R(R/m)$ and $d = \dim R$. Since E is Artinian, $F_R(E)$ is Artinian by Proposition 2.1(d). Hence, $F_R(E) \cong E^n$ for some integer $n \geq 1$. It suffices to show that $n = 1$. Let $U = U_R(R)$. By parts (d) and (f) of Proposition 2.3, we have an exact sequence $0 \longrightarrow R/U \longrightarrow H^d_m(\omega_R)^{\vee}$. Dualizing, we obtain a surjection $H^d_m(\omega_R) \longrightarrow E_{R/U} \longrightarrow 0$, where $E_{R/U} := E_{R/U}(R/m) \cong \operatorname{Hom}_R(R/U, E)$. Since ω_R is a finitely generated R-module, we have a surjection $R^s \longrightarrow \omega_R \longrightarrow 0$ for some s. This yields an exact sequence $H^d_m(R)^s \longrightarrow H^d_m(\omega_R) \longrightarrow 0$. Composing, we obtain an exact sequence

$$(*) \qquad H^d_m(R)^s \longrightarrow E_{R/U} \longrightarrow 0.$$

Now consider the exact sequence $0 \longrightarrow U \longrightarrow R \longrightarrow R/U \longrightarrow 0$. Applying the Matlis dual, we have that

$$0 \longrightarrow E_{R/U} \longrightarrow E \longrightarrow U^{\vee} \longrightarrow 0$$

is exact. Combining with (*), we obtain an exact sequence

$$H^d_m(R)^s \longrightarrow E \longrightarrow U^{\vee} \longrightarrow 0.$$

Applying F^e_R and using that $F_R(E) \cong E^n$, we obtain the exact sequence

$$H^d_m(R)^s \longrightarrow E^{n^e} \longrightarrow F^e_R(U^{\vee}) \longrightarrow 0.$$

Dualizing again, we have that

$$(**) \qquad 0 \longrightarrow F^e_R(U^{\vee})^{\vee} \longrightarrow R^{n^e} \longrightarrow \omega^s_R$$

is exact. Let $I = \operatorname{Ann}_R U = \operatorname{Ann}_R U^{\vee}$. Since $\dim R/I < d$ and $I^{[q]} \subseteq \operatorname{Ann}_R F^e_R(U^{\vee}) = \operatorname{Ann}_R F^e_R(U^{\vee})^{\vee}$ (where $q = p^e$), we have $\dim F^e(U^{\vee})^{\vee} < d$. Let P be a prime ideal of R such that $\dim R/P = d$. Localizing (**) at P, we obtain an injection $0 \longrightarrow R_P^{n^e} \longrightarrow \omega^s_{R_P}$. If $n > 1$, this is easily seen to be a contradiction by comparing lengths. □

We summarize some properties of FPI rings in the following proposition. Recall that a ring R is said to be *generically Gorenstein* if R_P is Gorenstein for every $P \in \operatorname{Min}_R R$.

Proposition 3.3. *Let R be a Noetherian ring.*

(a) If R is FPI and S is a multiplicatively closed set of R then R_S is FPI.

(b) If R is FPI then R is generically Gorenstein.

(c) If R is local then R is weakly FPI if and only if $\widehat{R}$ is weakly FPI.

(d) Let S be a faithfully flat R-algebra which is FPI and suppose that the fibers $k(P) \otimes_R S$ are generically Gorenstein for all $P \in \operatorname{Spec} R$. Then R is FPI.

(e) Suppose R is the homomorphic image of a Gorenstein local ring. If $\widehat{R}$ is FPI then so is R.

Proof. Part (a) follows easily from the fact that Frobenius commutes with localization. To prove (b), it suffices to prove that if R is local, zero-dimensional, and FPI then R is Gorenstein. In this situation, note that if M is a finitely generated R-module then $F^e_R(M)$

is free for sufficiently large e. Hence, as $F_R(E) \cong E$, E must be free. This implies R is injective and therefore Gorenstein.

Part (c) follows from the fact that $E_R(R/m) \cong E_{\widehat{R}}(\widehat{R}/\widehat{m})$.

To prove (d), let $P \in \operatorname{Spec} R$ and $E = E_R(R/P)$. It suffices to show that $F_R(E)$ is injective. Let $Q \in \operatorname{Spec} S$ which is minimal over PS. Then S_Q is a faithfully flat R_P-algebra and is FPI by part (b). Hence, we may assume (R, m) and (S, n) are local, $E = E_R(R/m)$, and the fiber S/mS is zero-dimensional Gorenstein. By [4, Theorem 1], $S \otimes_R E$ is injective. As S is FPI, $S \otimes_R F_R(E) \cong F_S(S \otimes_R E)$ is injective. Since S is faithfullly flat over R, this implies $F_R(E)$ is injective.

Part (e) follows from (d) since the hypothesis implies that the formal fibers of R are Gorenstein. □

The following result is essentially [10, Proposition 1.5]:

Proposition 3.4. *Let R be a quasi-Gorenstein local ring. Then R is FPI.*

Proof. Let $P \in \operatorname{Spec} R$ and $E = E_R(R/P)$. It suffices to show that $F_R(E)$ is injective. Since R_P is quasi-Gorenstein, we may assume $P = m$. Then $E \cong H^d_m(R)$ where $d = \dim R$. Hence, by Proposition 2.1(e), $F_R(E) \cong F_R(H^d_m(R)) \cong H^d_m(R) \cong E$. □

We next show that a weakly FPI ring has no embedded associated primes:

Proposition 3.5. *Let R be a weakly FPI ring. Then R satisfies Serre's condition S_1.*

Proof. Without loss of generality, we may assume that R is local and complete. Let $P \in \operatorname{Spec} R$ and $s = \dim R/P$. Since $\omega_{R/P}$ is a rank one torsion-free R/P-module, there exists an exact sequence $0 \longrightarrow \omega_{R/P} \longrightarrow R/P$. By Matlis duality, we have the exact sequence

$$E_{R/P} \longrightarrow H^s_m(R/P) \longrightarrow 0,$$

where $E_{R/P} := E_{R/P}(R/m)$. Applying F^e_R to this sequence, we obtain

$$F^e_R(E_{R/P}) \longrightarrow H^s_m(R/P^{[q]}) \longrightarrow 0$$

is exact, where $q = p^e$. By Proposition 2.3(a), $\operatorname{Ann}_R H^s_m(R/P^{[q]}) = U_R(R/P^{[q]})$. Note that as $\operatorname{Min}_R R/P^{[q]} = \{P\}$, $U_R(R/P^{[q]}) = \psi^{-1}(P^{[q]} R_P)$, where $\psi : R \longrightarrow R_P$ is the natural map.

Hence, for all $q = p^e$ we have

$$(\#) \qquad \operatorname{Ann}_R F_R^e(E_{R/P}) \subseteq \psi^{-1}(P^{[q]}R_P).$$

Now suppose that $P \in \operatorname{Ass}_R R$. Then there exists an exact sequence $0 \longrightarrow R/P \longrightarrow R$. Dualizing, we have $E \longrightarrow E_{R/P} \longrightarrow 0$ is exact, where $E = E_R(R/m)$. Applying F_R^e and using that $F_R(E) \cong E$, we have the exact sequence

$$E \longrightarrow F_R^e(E_{R/P}) \longrightarrow 0.$$

Dualizing again, we have that

$$0 \longrightarrow F_R^e(E_{R/P})^{\mathrm{v}} \longrightarrow R$$

is exact. Note that as $PE_{R/P} = 0$, $P^{[q]}F_R^e(E_{R/P}) = P^{[q]}F_R^e(E_{R/P})^{\mathrm{v}} = 0$ for all $q = p^e$. Hence, for all q we have an exact sequence

$$0 \longrightarrow F_R^e(E_{R/P})^{\mathrm{v}} \longrightarrow H_P^0(R).$$

Therefore, there exists a positive integer n such that $P^n \subseteq \operatorname{Ann}_R F_R^e(E_{R/P})$ for all e. By (#), this implies $P^n R_P \subseteq P^{[q]}R_P$ for all $q = p^e$. Hence, $P^n R_P = 0$ and $\operatorname{ht} P = 0$. $\square$

We remark that we do not know of a ring which is weakly FPI but not FPI. However, the two notions are equivalent for a large class of rings, as the following result shows:

Theorem 3.6. *Suppose R satisfies S_2 and is the homomorphic image of a Gorenstein ring. Then R is FPI if and only if R is weakly FPI.*

Proof. Assume R is weakly FPI. By Proposition 3.3(c) and (e), we can assume R is local and complete with canonical module ω_R. Note that $H_m^d(\omega_R)^{\mathrm{v}} \cong \operatorname{Hom}_R(\omega_R, \omega_R) \cong R$, the second isomorphism following from the fact that R is S_2. Hence, $H_m^d(\omega_R) \cong E$, where $E = E_R(R/m)$. Since $F_R(E) \cong E$ we have

$$E \cong F_R(E) \cong F_R(H_m^d(\omega_R)) \cong H_m^d(F_R(\omega_R)).$$

By local duality, this gives that $\operatorname{Hom}_R(F_R(\omega_R), \omega_R) \cong R$. Localizing at $P \in \operatorname{Spec} R$ we have $\operatorname{Hom}_{R_P}(F_{R_P}(\omega_{R_P}), \omega_{R_P}) \cong R_P$. Resetting notation (i.e., replacing R_P by R), it suffices to prove that if (R, m) is a local ring which is S_2, the homomorphic image of a Gorenstein ring, and with the property that $\operatorname{Hom}_R(F_R(\omega_R), \omega_R) \cong R$, then $F_R(E) \cong E$. Tensoring with $\widehat{R}$, we may assume R is complete. By local duality, we have that $H^d_m(F_R(\omega_R)) \cong E$. Since R is S_2, we have (as above) that $H^d_m(\omega_R) \cong E$. Thus,

$$F_R(E) \cong F_R(H^d_m(\omega_R)) \cong H^d_m(F_R(\omega_R)) \cong E.$$

□

In general, if R is weakly FPI and x is a non-zero-divisor on R then $R/(x)$ need not be weakly FPI. (Otherwise, using Propositions 3.3(b) and 3.5, one could prove that every weakly FPI ring is Gorenstein; but there exist quasi-Gorenstein rings which are not Gorenstein.) However, this does hold if in addition we have $\operatorname{Tor}_1^R(R^f, E) = 0$. We'll use this to give a criterion for R to be Gorenstein in terms of $\operatorname{Tor}_i^R(R^f, E)$. First, we prove the following lemma:

Lemma 3.7. *Let (R, m) be a local ring and I an ideal generated by a regular sequence. Then*

$$R/I \otimes_R E_{R/I^{[p]}}(R/m) \cong E_{R/I}(R/m).$$

Proof. Without loss of generality, we may assume R is complete. Let $E = E_R(R/m)$. Note that $E_{R/I^{[p]}}(R/m) \cong \operatorname{Hom}_R(R/I^{[p]}, E)$ and $E_{R/I}(R/m) \cong \operatorname{Hom}_R(R/I, E)$. Taking Matlis duals it suffices to prove that $\operatorname{Hom}_R(R/I, R/I^{[p]}) \cong R/I$. But this is easily seen to hold as I is generated by a regular sequence. □

The following result is in some sense 'dual' to Theorem 1.1 of [5], which holds in the case the Frobenius map is a finite morphism.

Theorem 3.8. *Let (R, m) be a local ring and $E = E_R(R/m)$. The following conditions are equivalent:*

(1) $F_R(E) \cong E$ and $\operatorname{Tor}_i^R(R^f, E) = 0$ for all $i = 1, \ldots, \operatorname{depth} R$;
(2) R is Gorenstein.

Proof. Condition (2) implies (1) by Lemma 2.2 and Proposition 3.4 (note $E \cong H^d_m(R)$). Conversely, suppose condition (1) holds. Let $\mathbf{x} = x_1, \ldots, x_r \in m$ be a maximal regular sequence on R and $K(\mathbf{x})$ the Koszul complex with respect to $\mathbf{x}$. Then $K(\mathbf{x}) \xrightarrow{\epsilon} R/(\mathbf{x}) \longrightarrow 0$ is exact, where ϵ is the augmentation map. Dualizing, we have $0 \longrightarrow E_{R/(\mathbf{x})}(R/m) \longrightarrow K(\mathbf{x})^{\vee}$

is exact. Since $K(\mathbf{x})_j^{\mathrm{v}} \cong E^{\binom{r}{j}}$ for all j and $\mathrm{Tor}_i^R(R^f, E) = 0$ for $1 \le i \le r$, we obtain that $0 \longrightarrow F_R(E_{R/(\mathbf{x})}(R/m)) \longrightarrow F_R(K(\mathbf{x})^{\mathrm{v}})$ is exact. In particular, since $F_R(E) \cong E$, we have that

$$0 \longrightarrow F_R(E_{R/(\mathbf{x})}(R/m)) \longrightarrow E \xrightarrow{[x_1^p \ldots x_r^p]} E^r$$

is exact. Hence,

$$F_R(E_{R/(\mathbf{x})}(R/m)) \cong \mathrm{Hom}_R(R/(\mathbf{x})^{[p]}, E) \cong E_{R/(\mathbf{x})^{[p]}}(R/m).$$

Using Lemma 3.7, we have

$$\begin{aligned} F_{R/(\mathbf{x})}(E_{R/(\mathbf{x})}(R/m)) &\cong R/(\mathbf{x}) \otimes_R F_R(E_{R/(\mathbf{x})}(R/m)) \\ &\cong R/(\mathbf{x}) \otimes_R E_{R/(\mathbf{x})^{[p]}}(R/m) \\ &\cong E_{R/(\mathbf{x})}(R/m). \end{aligned}$$

This says that $R/(\mathbf{x})$ is weakly FPI. Since $\operatorname{depth} R/(\mathbf{x}) = 0$, we must have $\dim R/(\mathbf{x}) = 0$ by Proposition 3.5. But then $R/(\mathbf{x})$ is Gorenstein by Proposition 3.3(b). Hence, R is Gorenstein. □

4. One-dimensional FPI rings

We now turn our attention to the one-dimensional case. If R is a local ring possessing an ideal which is also a canonical module of R, this ideal is referred to as a *canonical ideal* of R. If (R, m) is a one-dimensional Cohen-Macaulay local ring, then R has a canonical ideal (necessarily m-primary) if and only if $\widehat{R}$ is generically Gorenstein ([8, Satz 6.21]).

Theorem 4.1. *Let (R, m) be a one-dimensional local ring. The following conditions are equivalent:*

(a) R is weakly FPI;

(b) R is FPI;

(c) R is Cohen-Macaulay and has a canonical ideal ω_R such that $\omega_R \cong \omega_R^{[p]}$.

Proof. Since (b) trivially implies (a), it suffices to prove (a) implies (c) and (c) implies (b).

We first prove (a) implies (c): As R is weakly FPI, R is Cohen-Macaulay by Proposition 3.5. Furthemore, $\widehat{R}$ is weakly FPI and thus FPI by Theorem 3.6. Thus, $\widehat{R}$ is generically

Gorenstein, which implies R possesses a canonical ideal ω_R. To show $\omega_R \cong \omega_R^{[p]}$, it suffices to show that $\omega_R^{[p]}$ is a canonical ideal of R. Since $\omega_R^{[p]}$ is a canonical ideal for R if and only if $\omega_R^{[p]} \otimes_R \widehat{R} \cong (\omega_R \widehat{R})^{[p]}$ is a canonical ideal for $\widehat{R}$, we may assume without loss of generality that R is complete. Since R is Cohen-Macaulay, $H_m^1(\omega_R) \cong E$, where $E = E_R(R/m)$. Applying local cohomology to the exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow R/\omega_R \longrightarrow 0$$

we obtain

$$0 \longrightarrow R/\omega_R \longrightarrow E \longrightarrow H_m^1(R) \longrightarrow 0$$

is exact. Applying F_R and using that $\operatorname{Tor}_1^R(R^f, H_m^1(R)) = 0$ (Lemma 2.2), we have the exact sequence

$$0 \longrightarrow R/\omega_R^{[p]} \longrightarrow E \longrightarrow H_m^1(R) \longrightarrow 0.$$

Dualizing, we have

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow \operatorname{Hom}_R(R/\omega_R^{[p]}, E) \longrightarrow 0$$

is exact. But as $0 \longrightarrow \operatorname{Hom}_R(R/m, R/\omega_R^{[p]}) \longrightarrow \operatorname{Hom}_R(R/m, E)$ is exact, the socle of $R/\omega_R^{[p]}$ is one-dimensional and hence $R/\omega_R^{[p]}$ is Gorenstein. Thus, $\operatorname{Hom}_R(R/\omega_R^{[p]}, E) \cong R/\omega_R^{[p]}$ and we obtain an exact sequence

$$0 \longrightarrow \omega_R \longrightarrow R \longrightarrow R/\omega_R^{[p]} \longrightarrow 0.$$

This implies that $\omega_R \cong \omega_R^{[p]}$.

Next, we prove (c) implies (b): Since R is Cohen-Macaulay and possesses a canonical ideal, R is the homomorphic image of a Gorenstein ring (cf. [2, Theorem 3.3.6]). Hence, by Theorem 3.6, it suffices to prove that R is weakly FPI. Let $\pi : F_R(\omega_R) \longrightarrow \omega_R^{[p]}$ be the natural surjection given by $\pi(r \otimes u) = ru^p$ and let $C = \ker \pi$. Since R_P is Gorenstein for all primes $P \neq m$, $\dim C = 0$. Consequently, $H_m^1(C) = 0$ and $H_m^1(F_R(\omega_R)) \cong H_m^1(\omega_R^{[p]})$. Since $E \cong H_m^1(\omega_R)$ and $\omega_R \cong \omega_R^{[p]}$, we have

$$F_R(E) \cong F_R(H_m^1(\omega_R)) \cong H_m^1(F_R(\omega_R)) \cong H_m^1(\omega_R^{[p]}) \cong H_m^1(\omega_R) \cong E.$$

Hence, R is weakly FPI. $\square$

We remark that Theorem 4.1 is dual to Lemma 2.6 of [5] in the case Frobenius map is finite. In particular, since condition (c) appears in both results, we have the following:

Corollary 4.2. *Let (R, m) be a local ring of dimension at most one such that the Frobenius map is finite and set $E = E_R(R/m)$. Let $\operatorname{Hom}_R(R^f, R)$ denote the set of right R-module homomorphisms from R^f to R and view $\operatorname{Hom}_R(R^f, R)$ as a left R-module in the natural way. The following are equivalent:*

(a) $\operatorname{Hom}_R(R^f, R) \cong R$*;*

(b) $R^f \otimes_R E \cong E$*.*

Proof. If $\dim R = 0$, it is easily seen that both conditions are equivalent to R being Gorenstein. In the case that $\dim R = 1$, combine Theorem 4.1 and [5, Lemma 2.6]. □

It would be interesting to know whether the above corollary holds in dimensions greater than one, at least in the case R is Cohen-Macaulay. Note that in general the modules $\operatorname{Hom}_R(R^f, R)$ and $R^f \otimes_R E$ do not form a Matlis pair, even in the case R is Cohen-Macaulay and F-finite.

We remark that there exists one-dimensional local FPI rings which are not Gorenstein. In fact, the next result shows that every one-dimensional F-pure ring is FPI. Recall that a homomorphism $A \longrightarrow B$ of commutative rings is called *pure* if the map $M \longrightarrow B \otimes_A M$ is injective for every A-module M. A ring R of prime characteristic is called *F-pure* if the Frobenius map $f : R \longrightarrow R$ is pure.

Proposition 4.3. *Let (R, m) be a one-dimensional F-pure ring. Then R is FPI.*

Proof. We may assume R is local. By Theorem 4.1, it suffices to show that R is weakly FPI. By Proposition 3.3(c) and [9, Corollary 6.13], we may assume R is complete. Let k be the residue field of R. By the Cohen Structure Theorem, $R \cong A/I$ where $A = k[[T_1, \ldots, T_n]]$ and $T_1, \ldots, T_n$ are indeterminates. Let ℓ be the algebraic closure k, $B = \ell[[T_1, \ldots, T_n]]$, and $S = B/IB$. Note that as B is faithfully flat over A, S is faithfully flat over R. Furthermore, S is F-pure since R is (e.g., [3, Theorem 1.12]). Finally, by [4, Theorem 1], $E_S(\ell) \cong E_R(k) \otimes_R S$. Hence, S is weakly FPI if and only if R is weakly FPI. Thus, resetting notation, we may assume R is complete and its residue field k is algebraically closed. By [6, Theorem 1.1], $R \cong k[[T_1, \dots, T_n]]/I$ where $I = (\{T_iT_j \mid 1 \le i < j \le n\})$. It is easily checked that $\omega_R = (T_2 - T_1, \dots, T_n - T_1)R$ is a canonical ideal of R and that $\omega_R^{[p]} = (T_1 + \cdots + T_n)^{p-1}\omega_R$ (cf. [5, Example 2.8]). Hence, $\omega_R^{[p]} \cong \omega_R$ and R is weakly FPI by Theorem 4.1. □

As a specific example of a one-dimensional non-Gorenstein FPI ring, let k be any field of characteristic p and $R = k[[x, y, z]]/(xy, xz, yz)$. Then R is a one-dimensional local ring which is F-pure (and hence FPI) but not Gorenstein. Notice in this example that R has three associated primes. Regarding this we note the following, which is a consequence of Corollary 1.3 of [5]:

Corollary 4.4. *Let R be a one-dimensional complete local ring with algebraically closed residue field and suppose R has at most two associated primes. The following are equivalent:*

(a) R is weakly FPI;

(b) R is Gorenstein.

Proof. Notice that the hypotheses imply that R is F-finite. Hence, (a) is equivalent to the condition that $\operatorname{Hom}_R(R^f, R) \cong R$ by Corollary 4.2. The result now folllows from [5, Corollary 1.3]. □

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