THE FROBENIUS FUNCTOR AND INJECTIVE MODULES

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Abstract. We investigate commutative Noetherian rings of prime characteristic such that the Frobenius functor applied to any injective module is again injective. We characterize the class of one-dimensional local rings with this property and show that it includes all one-dimensional $F$-pure rings. We also give a characterization of Gorenstein local rings in terms of $\text{Tor}_i^R(R^f, E)$, where $E$ is the injective hull of the residue field and $R^f$ is the ring $R$ whose right $R$-module action is given by the Frobenius map.

1. Introduction

Let $R$ be a commutative Noetherian ring of prime characteristic $p$ and $f : R \rightarrow R$ the Frobenius ring homomorphism (i.e., $f(r) = r^p$ for $r \in R$). We let $R^f$ denote the ring $R$ with the $R$-$R$ bimodule structure given by $r \cdot s := rs$ and $s \cdot r := sf(r)$ for $r \in R$ and $s \in R^f$. Then $F_R(\cdot) := R^f \otimes_R \cdot$ is a right exact functor on the category of (left) $R$-modules and is called the Frobenius functor on $R$. This functor has played an essential role in the solution of many important problems in commutative algebra for local rings of prime characteristic (e.g., Hochster and J. Roberts [11], Peskine and Szpiro [16], P. Roberts [17]). Of particular interest is how properties of the Frobenius map (or functor) characterize classical properties of the ring. The most important result of this type, proved by Kunz [13], says that $F_R$ is exact if and only if $R$ is a regular ring. As another example, Iyengar and Sather-Wagstaff prove that a local ring $R$ is Gorenstein if and only if $R^f$ (viewed as a right $R$-module) has finite G-dimension [14, Theorems 6.2 and 6.6].

As $F_R$ is additive and $F_R(R) \cong R$, it is easily seen that $F_R$ preserves projective (in fact, flat) modules. In this paper, we consider rings for which $F_R$ preserves injective modules, i.e., rings $R$ having the property that $F_R(I)$ is injective for every injective $R$-module $I$. A result of Huneke and Sharp [12, Lemma 1.4] shows that Gorenstein rings have this property, and in fact this is true for quasi-Gorenstein rings as well (Proposition 3.6). In Section 3, we show that if $F_R$ preserves injectives then $R$ satisfies Serre’s condition $S_1$ and that $F_R(I) \cong I$ for every injective $R$-module $I$. Moreover, if $R$ is a homomorphic image of a Gorenstein local ring, then $F_R$ preserves all injectives if and only if $F_R(E)$ is injective, where $E$ is the injective hull of the residue field. We also give a criterion (Theorem 3.14) for a local ring $R$ to be Gorenstein in terms of $\text{Tor}_1^R(R^f, E)$:

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Theorem 1.1. Let \((R, m)\) be a local ring and \(E = E_R(R/m)\). Then the following are equivalent:

(a) \(R\) is Gorenstein;
(b) \(\text{Tor}_0^R(R^f, E) \cong E\) and \(\text{Tor}_i^R(R^f, E) = 0\) for \(i = 1, \ldots, \text{depth } R\).

In Section 4, we study one-dimensional rings \(R\) such that \(F_R\) preserves injectives. In particular, we give the following characterization (Theorem 4.1) in the case where \(R\) is local:

Theorem 1.2. Let \((R, m)\) be a one-dimensional local ring and \(E = E_R(R/m)\). The following conditions are equivalent:

(a) \(F_R(E)\) is injective;
(b) \(F_R(I) \cong I\) for all injective \(R\)-modules \(I\);
(c) \(R\) is Cohen-Macaulay and has a canonical ideal \(\omega_R\) such that \(\omega_R \cong \omega_R^{[p]}\).

Using this characterization, we show that every one-dimensional \(F\)-pure ring preserves injectives. We also prove, using a result of Goto [7], that if \(R\) is a complete one-dimensional local ring with algebraically closed residue field and has at most two associated primes then \(R\) is Gorenstein if and only if \(F_R\) preserves injectives. We remark that Theorems 1.1 and 1.2 are dual to results appearing in [7] in the case where the Frobenius map is finite. This duality is made explicit in Proposition 3.10.

In Section 2, we summarize several results concerning the Frobenius functor and canonical modules which will be needed in the later sections. Most of these are well-known, but for some we could not find a reference in the literature.

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2. Some properties of the Frobenius functor and canonical modules

Throughout this paper \(R\) denotes a commutative Noetherian ring of prime characteristic \(p\). For an \(R\)-module \(M\), \(E_R(M)\) will denote the injective hull of \(M\). If \(I\) is an ideal of \(R\) then \(H_I^j(M)\) will denote the \(j\)th local cohomology module of \(M\) with support in \(I\). If \(R\) is local with maximal ideal \(m\), we denote the \(m\)-adic completion of \(R\) by \(\hat{R}\). We refer the reader to [2] or [15] for any unexplained terminology or notation.

Let \(M\) be a finitely generated \(R\)-module with presentation \(R^r \xrightarrow{\varphi} R^s \rightarrow M \rightarrow 0\), where \(\varphi\) is represented (after fixing bases) by an \(s \times r\) matrix \(A\). Then \(F_R(M)\) has the presentation \(R^r \xrightarrow{F_R(\varphi)} R^s \rightarrow F_R(M) \rightarrow 0\) and the map \(F_R(\varphi)\) is represented by the matrix \(A^{[p]}\) obtained by raising the corresponding entries of \(A\) to the \(p\)th power. For an ideal \(I\) of \(R\) and \(q = p^e\), we let \(I^{[q]}\) denote the ideal generated by the set \(\{i^q \mid i \in I\}\). Note that, by above, \(F_R^e(R/I) \cong R/I^{[q]}\), where \(F_R^e\) is the functor \(F_R\) iterated \(e\) times.

The following proposition lists a few properties of the Frobenius functor which we will use in the sequel. Most of these are well known:
Proposition 2.1. Let $M$ be an $R$-module and $S$ a multiplicatively closed set of $R$.

(a) If $T$ is an $R$-algebra then there is a $T$-isomorphism $F_T(T \otimes_R M) \cong T \otimes_R F_R(M)$. In particular, $F_R(M)_S \cong F_R(S)(M_S)$ as $R_S$-modules.

(b) For any $R_S$-module $N$, the map $h : F_R(N) \rightarrow F_{R_S}(N)$ given by $h(r \otimes n) = \frac{r}{1} \otimes n$ for $r \in R$ and $n \in N$ is an $R_S$-isomorphism.

(c) $\text{Supp}_{R} F_R(M) = \text{Supp}_{R_S} F_{R_S}(M)$.

(d) $M$ is Artinian if and only if $F_R(M)$ is Artinian.

(e) If $(R, m)$ is local, and $M$ is finitely generated of dimension $s$, then $F_R(H^s_m(M)) \cong H^s_m(F_R(M))$.

Proof. Part (a) follows easily from properties of tensor products. For part (b), one first observes that the map $g : R^f \otimes_R R_S \rightarrow (R_S)^f$ given by $g(r \otimes 2) = r \frac{a}{a^p}$ is an isomorphism of $R_S$-$R_S$ bimodules. Tensoring with $N$ (over $R_S$) on the right gives the desired isomorphism. For (c), it suffices to show $\text{Supp}_{R} F_R(M) \subseteq \text{Supp}_{R_S} F_{R_S}(M)$. By part (a), it is enough to prove that if $R$ is a complete local ring and $M \neq 0$ then $F_R(M) \neq 0$. Then $R = S/I$ where $S$ is a regular local ring of characteristic $p$. Hence, we may assume that $R$ is complete. Then $R = S/I$ where $S$ is a regular local ring of characteristic $p$. As $F_R(M) \cong S/I \otimes_S F_S(M)$, it is clear that if $F_S(M)$ is Artinian then so is $F_R(M)$. Conversely, if $S/I \otimes_S F_S(M)$ is Artinian, then $F_S(M)$ is Artinian since we also have $I[p]F_S(M) = 0$. Thus, it is enough to prove the result in the case where $R$ is a regular local ring. Recall that an $R$-module is Artinian if and only if $\text{Supp}_{R} N \subseteq \{m\}$ and $(0 :_M m)$ is finitely generated. Since $\text{Supp}_{R} M = \text{Supp}_{R_S} F_R(M)$, it suffices to prove that $(0 :_M m)$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated. As $F_R$ is exact, $F_R((0 :_M m)) \cong (0 :_{F_R(M)} m)^{[p]}$. But $(0 :_M m)$ is finitely generated if and only if $F_R((0 :_M m))$ is finitely generated, and $(0 :_{F_R(M)} m)^{[p]}$ is finitely generated if and only if $(0 :_{F_R(M)} m)$ is finitely generated.

For (e), let $I = \text{Ann}_R M$ and choose $x_1, \ldots, x_s \in m$ such that their images in $R/I$ form a system of parameters. Set $J = (x_1, \ldots, x_s)$. Then $H^s_m(M) \cong H^s_J(M)$. Since $H^i_J(R) = 0$ for all $i > s$, $T \otimes_R H^s_J(M) \cong H^s_J(T \otimes_R M)$ for any $R$-algebra $T$. Then

$$F_R(H^s_m(M)) \cong R^f \otimes_R H^s_J(M) \cong H^s_J[F_R(M)].$$

Finally, as $I^{[p]} \subseteq \text{Ann}_R F_R(M)$ and $J^{[p]} + I^{[p]}$ is $m$-primary, we have $H^s_J[F_R(M)] \cong H^s_m(F_R(M))$. □

We need one more (likely well-known) result concerning the Frobenius:
Lemma 2.2. Let \((R, m)\) be a local ring of dimension \(d\). If \(R\) is Cohen-Macaulay then for all \(i \geq 1\) we have \(\Tor^R_i(R^f, H^d_m(R)) = 0\).

Proof. Let \(x = x_1, \ldots, x_d\) be a system of parameters for \(R\) and \(C(x)\) the Čech cochain complex with respect to \(x\). Note that \(F_R(C(x)) \cong C(x^p)\) where \(x^p = x_1^p, \ldots, x_d^p\). Since \(R\) is Cohen-Macaulay, \(x\) is a regular sequence and thus \(C(x)\) is a flat resolution of \(H^d_m(R)\).

Hence for \(i \geq 1\),
\[
\Tor^R_i(R^f, H^d_m(R)) \cong H^{d-i}(R^f \otimes_R C(x)) \cong H^{d-i}(C(x^p)) = 0.
\]

\(\square\)

For a nonzero finitely generated \(R\)-module \(M\) we let \(U_R(M)\) be the intersection of all the primary components \(Q\) of 0 in \(M\) such that \(\dim M/Q = \dim M\). It is easily seen that \(U_R(M) = \{x \in M \mid \dim Rx < \dim M\}\). A local ring \(R\) is said to be quasi-unmixed if \(U_R(\hat{R}) = 0\).

Let \((R, m)\) be a local ring of dimension \(d\), \(E = E_R(R/m)\), and \((-)^\vee := \Hom_R(-, E)\) the Matlis duality functor. A finitely generated \(R\)-module \(K\) is called a canonical module of \(R\) if there is an \(\hat{R}\)-isomorphism \(K \otimes_R \hat{R} \cong H^d_m(R)^\vee\). If a canonical module exists, it is unique up to isomorphism and denoted by \(\omega_R\). Any complete local ring possesses a canonical module.

More generally, \(R\) possesses a canonical module if \(R\) is a homomorphic image of a Gorenstein ring. Proofs of these facts can be found in Aoyama [1] (or the references cited there). We summarize some additional properties of canonical modules in the following proposition:

Proposition 2.3. Let \(R\) be a local ring which possesses a canonical module \(\omega_R\) and let \(h : \hat{R} \rightarrow \Hom_R(\omega_R, \omega_R)\) be the natural map. The following hold:

(a) \(\Ann_R \omega_R = U_R(R)\).
(b) \((\omega_R)_P \cong \omega_{R_P}\) for every prime \(P \in \Supp_R \omega_R\).
(c) \(\omega_R \otimes_R \hat{R} \cong \omega_{\hat{R}}\).
(d) \(\ker h = U_R(\hat{R})\).
(e) \(h\) is an isomorphism if and only if \(R\) satisfies Serre’s condition \(S_2\).
(f) If \(R\) is complete, \(\Hom_R(M, \omega_R) \cong H^d_m(M)^\vee\) for any \(R\)-module \(M\).

Proof. The proofs of parts (a)-(e) can be found in [1]. Part (f) is just local duality, but it can also be seen directly from the definition of \(\omega_R\) and adjointness:
\[
\Hom_R(H^d_m(M), E) \cong \Hom_R(M \otimes_R H^d_m(R), E) \cong \Hom_R(M, H^d_m(R)^\vee).
\]

\(\square\)

If \(R\) is a local ring possessing a canonical module \(\omega_R\) such that \(\omega_R \cong R\), then \(R\) is said to be quasi-Gorenstein. Equivalently, \(R\) is quasi-Gorenstein if and only if \(H^d_m(R) \cong E\). By the proposition above, if \(R\) is quasi-Gorenstein then \(R\) is \(S_2\) and \(R_P\) is quasi-Gorenstein for every \(P \in \Spec R\). It is easily seen that \(R\) is Gorenstein if and only if \(R\) is Cohen-Macaulay and quasi-Gorenstein. Finally, there exist quasi-Gorenstein rings which are not Cohen-Macaulay. Examples of such rings can be constructed using Theorem 2.11 and Corollary 2.12 of [1].
3. Rings for which Frobenius Preserves Injectives

To facilitate our discussion we make the following definition:

**Definition 3.1.** The ring \( R \) is said to be FPI (i.e., `Frobenius Preserves Injectives`) if \( F_R(I) \) is injective for every injective \( R \)-module \( I \). We say that \( R \) is weakly FPI if \( F_R(I) \) is injective for every Artinian injective \( R \)-module \( I \).

By Matlis’s decomposition theory for injective modules ([15, Theorems 18.4 and 18.5]), every injective \( R \)-module is a direct sum of modules of the form \( E_R(R/P) \) for various prime ideals \( P \) of \( R \). Further, \( E_R(R/P) \) is Artinian if and only if \( P \) is a maximal ideal of \( R \). Accordingly, we see that \( R \) is FPI (respectively, weakly FPI) if and only if \( F_R(E_R(R/P)) \) is injective for every prime (respectively, maximal) ideal \( P \) of \( R \).

**Lemma 3.2.** Let \( P \) be a prime ideal of \( R \) and \( S \) a multiplicatively closed set of \( R \) such that \( S \cap P = \emptyset \). Then the following hold:

\begin{enumerate}[(a)]
    \item \( F_R(E_R(R/P)) \cong F_{R_S}(E_{R_S}(R_S/PR_S)) \) as \( R_S \)-modules.
    \item \( F_R(E_R(R/P)) \) is injective as an \( R \)-module if and only if \( F_{R_S}(E_{R_S}(R_S/PR_S)) \) is injective as an \( R_S \)-module.
\end{enumerate}

**Proof.** To prove (a), note that by Lemma 10.1.12 and Proposition 10.1.13 of [3], there are \( R_S \)-isomorphisms \( E_R(R/P) \cong E_R(R/P)_S \cong E_{R_S}(R_S/PR_S) \). Hence, by Proposition 2.1(b) we have that \( F_R(E_R(R/P)) \cong F_{R_S}(E_{R_S}(R_S/PR_S)) \) as \( R_S \)-modules. Part (b) follows from (a) and the fact that an \( R_S \)-module is injective as an \( R \)-module if and only if it is injective as an \( R_S \)-module ([3, Lemma 10.1.11]). \( \square \)

Combining this Lemma with the remarks in the previous paragraph, we obtain the following:

**Proposition 3.3.** The following hold for the ring \( R \):

\begin{enumerate}[(a)]
    \item \( R \) is weakly FPI if and only if \( R_m \) is weakly FPI for every maximal ideal \( m \) of \( R \).
    \item \( R \) is FPI if and only if \( R_m \) is FPI for every maximal ideal \( m \) of \( R \).
    \item \( R \) is FPI if and only if \( R_P \) is weakly FPI for every prime ideal \( P \) of \( R \).
\end{enumerate}

**Proof.** We have that \( R \) is weakly FPI if and only if \( F_R(E_R(R/m)) \) is injective for every maximal ideal \( m \) of \( R \). For a given maximal ideal \( m \), we have by part (b) of Lemma 3.2 that \( F_R(E_R(R/m)) \) is injective if and only if \( F_{R_m}(E_{R_m}(R/m)) \) is injective as an \( R_m \)-module. As the latter condition holds if and only if \( R_m \) is weakly FPI, we see that (a) holds. Parts (b) and (c) are proved similarly. \( \square \)

We summarize some properties of FPI rings in the following proposition. Recall that a ring \( R \) is said to be generically Gorenstein if \( R_P \) is Gorenstein for every \( P \in \text{Min}_R R \).

**Proposition 3.4.** Let \( R \) be a Noetherian ring.

\begin{enumerate}[(a)]
    \item If \( R \) is FPI and \( S \) is a multiplicatively closed set of \( R \) then \( R_S \) is FPI.
    \item If \( R \) is FPI then \( R \) is generically Gorenstein.
    \item If \( R \) is local then \( R \) is weakly FPI if and only if \( \hat{R} \) is weakly FPI.
    \item Let \( S \) be a faithfully flat \( R \)-algebra which is FPI and suppose that the fibers \( k(P) \otimes_R S \) are generically Gorenstein for all \( P \in \text{Spec} \ R \). Then \( R \) is FPI.
    \item Suppose \( R \) is a quotient of a Gorenstein local ring. If \( \hat{R} \) is FPI then so is \( R \).
\end{enumerate}
Proof. Part (a) follows from Lemma 3.2(b). To prove (b), let \( P \in \text{Min}_R R \). By part (a), \( R_P \) is FPI. Resetting notation, it suffices to show that if \( (R, m) \) is a zero-dimensional local FPI ring then \( R \) is Gorenstein. In this situation, note that if \( M \) is a finitely generated \( R \)-module then \( F^e_R(M) \) is free for sufficiently large \( e \). To see this, consider a minimal presentation

\[
\begin{array}{c}
  R^e \to R^t \to M \to 0,
\end{array}
\]

where \( A \) is a matrix all of whose entries are in \( m \). Applying \( F^e_R(-) \) to this presentation and noting that \( A[q] = 0 \) for sufficiently large \( q = p^e \) (since \( m \) is nilpotent), we obtain that \( F^e_R(M) \cong R^t \) for sufficiently large \( e \). Now let \( E = E_R(R/m) \), which is a finitely generated \( R \)-module. Since \( R \) is FPI, \( F^e_R(E) \) is injective for all \( e \). Also, \( F^e_R(E) \neq 0 \) for all \( e \) by Proposition 2.1(c). Hence, there exists a nonzero free \( R \)-module which is injective. This implies \( R \) is injective and therefore Gorenstein.

For part (c), let \( E = \hat{E}_R(R/m) \). Then \( E \cong \hat{R} \otimes_R E \cong \hat{E}_R(\hat{R}/\hat{m}) \) as \( \hat{R} \)-modules. By Proposition 2.1(d), \( F_R(E) \) is Artinian and consequently \( \hat{R} \otimes_R F_R(E) \cong F_R(E) \). By part (a) of Proposition 2.1, we have \( \hat{R} \)-isomorphisms \( F_R(E) \cong \hat{R} \otimes_R F_R(E) \cong F_R(E) \). Now, \( F_R(E) \) is \( R \)-injective if and only if there is an \( R \)-isomorphism \( F_R(E) \cong E^n \) for some \( n \). However, every \( R \)-homomorphism between Artinian modules is also an \( \hat{R} \)-homomorphism. Hence, \( F_R(E) \cong E^n \) if and only if \( F^e_R(E) \cong E^n \) as \( \hat{R} \)-modules. Thus, \( F_R(E) \) is \( R \)-injective if and only if \( F^e_R(E) \) is \( \hat{R} \)-injective.

To prove (d), let \( P \) be an arbitrary prime ideal of \( R \). By Proposition 3.3(c), it suffices to show that \( R_P \) is weakly FPI. Let \( Q \in \text{Spec} S \) which is minimal over \( PS \). Then \( S_Q \) is a faithfully flat \( R_P \)-algebra and is FPI by part (a). Hence, we may assume \( (R, m) \) and \( (S, n) \) are local, \( P = m \), and the fiber \( S/mS \) is zero-dimensional Gorenstein. Let \( E = E_R(R/m) \).

By \([6, \text{Theorem } 1]\), \( S \otimes_R E \) is an injective \( S \)-module. As \( S \) is FPI, \( S \otimes_R F_R(E) \cong F_S(S \otimes_R E) \) is injective. Since \( S \) is faithfully flat over \( R \), this implies \( F_R(E) \) is an injective \( R \)-module.

Part (e) follows from (d) since the hypothesis implies that the formal fibers of \( R \) are Gorenstein (see \([15, \text{Exercise } 23.1]\)).

We now show that \( F_R \) preserves injectives if and only if \( F_R \) fixes injectives:

**Proposition 3.5.** Let \( I \) be an injective \( R \)-module and suppose \( F_R(I) \) is injective. Then \( F_R(I) \cong I \).

*Proof.* By Matlis’s decomposition theorem, it suffices to prove the result in the case when \( I = E_R(R/P) \) where \( P \) is a prime ideal. By Lemma 3.2(a), we may assume \( (R, m) \) is local and \( P = m \). Furthermore, by the arguments used in the proof of Proposition 3.4(c), we may also assume \( R \) is complete. Let \( E = E_R(R/m) \) and \( d = \text{dim} R \). Since \( E \) is Artinian, \( F_R(E) \) is Artinian by Proposition 2.1(d). Hence, \( F_R(E) \cong E^n \) for some integer \( n \geq 1 \). It suffices to show that \( n = 1 \). Let \( U = U_R(R) \). By parts (d) and (f) of Proposition 2.3, we have an exact sequence \( 0 \to R/U \to H^d_m(\omega_R)^\vee \). Dualizing, we obtain a surjection \( H^d_m(\omega_R)^\vee \to E_{R/U} \), where \( E_{R/U} := E_{R/U}(R/m) \cong \text{Hom}_R(R/U, E) \). Since \( \omega_R \) is a finitely generated \( R \)-module, we also have a surjection \( R^s \to \omega_R \) for some \( s \). This yields an exact sequence \( H^d_m(R)^s \to H^d_m(\omega_R) \to 0 \). Composing, we obtain an exact sequence

\[
(*)
H^d_m(R)^s \to E_{R/U} \to 0.
\]

Now consider the short exact sequence \( 0 \to U \to R \to R/U \to 0 \). Applying Matlis duality, we have that

\[
0 \to E_{R/U} \to E \to U^\vee \to 0
\]
is exact. Combining with (*), we have an exact sequence

\[ H^d_m(R)^s \rightarrow E \rightarrow U \rightarrow 0. \]

Applying \( F_R(\cdot) \) and using that \( F_R(E) \cong E^n \) and \( F_R(H^d_m(R)) \cong H^d_m(R) \), we obtain an exact sequence

\[ H^d_m(R)^s \rightarrow E^{ns} \rightarrow F_R(U^\vee) \rightarrow 0. \]

Dualizing once again yields an exact sequence

\[ (**): 0 \rightarrow F_R(U^\vee)^y \rightarrow R^{ne} \rightarrow (\omega_R)^s. \]

Let \( I = \text{Ann}_R U = \text{Ann}_R U^\vee \). Since \( \dim R/I < d \) and \( I^{[q]} \subseteq \text{Ann}_R F_R(U^\vee) = \text{Ann}_R F_R(U^\vee)^y \) (where \( q = p^f \)), we have \( \dim F_R(U^\vee)^y < d \). Let \( P \) be a prime ideal of \( R \) such that \( \dim R/P = d \). Localizing (**) at \( P \), we have the exactness of \( 0 \rightarrow R_P^{ne} \rightarrow (\omega_{RP})^s \). If \( n > 1 \), we easily obtain a contradiction by comparing lengths, as \( e \) can be arbitrarily large. \( \square \)

The following result is essentially [12, Proposition 1.5]:

**Proposition 3.6.** Let \( (R, m) \) be a quasi-Gorenstein local ring. Then \( R \) is FPI.

**Proof.** It suffices to prove that \( R_P \) is weakly FPI for every prime ideal \( P \). As \( R_P \) is quasi-Gorenstein, we may assume \( P = m \). Let \( E = E_R(R/m) \). Then \( E \cong H^d_m(R) \) where \( d = \dim R \). Hence, by Proposition 2.1(e), \( F_R(E) \cong F_R(H^d_m(R)) \cong H^d_m(R) \cong E \). \( \square \)

Next, we show that for a large class of rings, weakly FPI implies FPI. We first establish a few preliminary results.

**Lemma 3.7.** Let \( \varphi : R \rightarrow S \) be a homomorphism of local rings such that \( S \) is finitely generated as an \( R \)-module. Let \( k \) and \( \ell \) denote the residue fields of \( R \) and \( S \), respectively, and set \( E_R = E_R(k) \) and \( E_S = E_S(\ell) \). Then \( \text{Hom}_R(S, E_R) \cong E_S \) as \( S \)-modules.

**Proof.** Let \( m \) and \( n \) be the maximal ideals of \( R \) and \( S \), respectively. As \( S \) is finitely generated over \( R \), \( \varphi^{-1}(n) = m \) and \( \ell \cong k^t \) as \( R \)-modules for some positive integer \( t \). Clearly, \( \text{Hom}_R(S, E_R) \) is injective as an \( S \)-module. Also \( \text{Supp}_S \text{Hom}_R(S, E_R) = \{ m \} \), since \( \text{Supp}_R E_R = \{ m \} \). Hence, \( \text{Hom}_R(S, E_R) \cong (E_S)^r \) where \( r = \dim_{\ell} \text{Hom}_S(\ell, \text{Hom}_R(S, E_R)) \). It suffices to show that \( r = 1 \). We have the following \( R \)-module isomorphisms:

\[ \text{Hom}_S(\ell, \text{Hom}_R(S, E_R)) \cong \text{Hom}_R(\ell \otimes_S S, E_R) \cong \text{Hom}_R(k^t, E_R) \cong k^t. \]

Thus, \( r = \dim_{\ell} \text{Hom}_S(\ell, \text{Hom}_R(S, E_R)) = \frac{1}{t} \dim_{k} \text{Hom}_S(\ell, \text{Hom}_R(S, E_R)) = 1 \). \( \square \)

**Proposition 3.8.** Let \( \varphi : R \rightarrow S \) be a homomorphism of local rings such that \( S \) is finitely generated as an \( R \)-module. Let \( E_R \) and \( E_S \) be as in Lemma 3.7. Then for any \( R \)-module \( M \) we have an \( S \)-module isomorphism

\[ \text{Hom}_S(S \otimes_R M, E_S) \cong \text{Hom}_R(S, \text{Hom}_R(M, E_R)). \]
Proof. Using adjointness and Lemma 3.7, we have $S$-module isomorphisms

$$\text{Hom}_S(S \otimes_R M, E_S) \cong \text{Hom}_S(S \otimes_R M, \text{Hom}_R(S, E_R))$$

$$\cong \text{Hom}_R(S \otimes_R M, E_R)$$

$$\cong \text{Hom}_R(S, \text{Hom}_R(M, E_R)).$$

Given an arbitrary $R$-module $M$, we let $\text{Hom}_R(R^f, M)$ denote the abelian group of right $R$-module homomorphisms from $R^f$ to $M$. (As $R$ is commutative, we can conveniently view $M$ as either a left or right $R$-module as the situation warrants.) Note that this group has a natural structure as a left $R$-module via the left $R$-action on $R^f$; i.e., $rh(s) = h(rs)$ for all $r \in R, s \in R^f$, and $h \in \text{Hom}_R(R^f, M)$.

Recall that a ring $R$ of characteristic $p$ is called $F$-finite if the Frobenius map $f : R \to R$ is a finite morphism; i.e., $R^f$ is finitely generated as a right $R$-module. If, in Proposition 3.8, we take $S$ to be $R$ and $\varphi$ to be the Frobenius homomorphism $f$, we obtain the following:

**Corollary 3.9.** Let $R$ be an $F$-finite local ring. For any $R$-module $M$ we have an isomorphism of left $R$-modules

$$F_R(M^\vee) \cong \text{Hom}_R(R^f, M^\vee).$$

**Proposition 3.10.** Suppose $R$ is an $F$-finite local ring and $E$ the injective hull of the residue field of $R$. The following are equivalent:

(a) $\text{Hom}_R(R^f, R) \cong R$;

(b) $F_R(E) \cong E$.

**Proof.** We first claim that it suffices to prove the result in the case when $R$ is complete. By Propositions 3.4(c) and 3.5, we have that $F_R(E) \cong E$ as $R$-modules if and only if $F_{\widehat{R}}(E) \cong E$ as $\widehat{R}$-modules. Let $S$ be the ring $R$ considered as an $R$-module via $f$, and let $T$ be the ring $\widehat{R}$ viewed as an $\widehat{R}$-module via $f$. Since $S$ is finitely generated as an $R$-module, $\widehat{R} \otimes_R S \cong \widehat{S} = T$ as $\widehat{R}$-modules. Thus, $\widehat{S} \otimes_{\widehat{R}} \text{Hom}_{\widehat{R}}(S, R) \cong \widehat{R} \otimes_{\widehat{R}} \text{Hom}_{\widehat{R}}(S, R) \cong \text{Hom}_{\widehat{R}}(T, \widehat{R})$. Since $\text{Hom}_{\widehat{R}}(S, R)$ is finitely generated as an $S$-module, $\text{Hom}_{\widehat{R}}(S, R) \cong S$ as $S$-modules if and only if $\text{Hom}_{\widehat{R}}(T, \widehat{R}) \cong T$ as $T$-modules ([5, Exercise 7.5]). In other words, $\text{Hom}_{\widehat{R}}(R^f, R) \cong R$ if and only if $\text{Hom}_{\widehat{R}}(\widehat{R}^f, \widehat{R}) \cong \widehat{R}$. This proves the claim.

Now suppose $\widehat{R}$ is complete. By Corollary 3.9, we have $F_{\widehat{R}}(E)^\vee \cong \text{Hom}_{\widehat{R}}(R^f, R)$. The result now follows by Matlis Duality.

**Theorem 3.11.** Let $R$ be a homomorphic image of a Gorenstein ring. Then $R$ is FPI if and only if $R$ is weakly FPI.

**Proof.** By parts (a) and (b) of Proposition 3.3, it suffices to prove this in the case when $R$ is a local ring with maximal ideal $m$. We first prove the theorem in the case when $R$ is $F$-finite. Suppose $F_R(E) \cong E$ and let $P \in \text{Spec} R$. By Proposition 3.10, we have $\text{Hom}_R(R^f, R) \cong R$. As $R^f$ is finitely generated as a right $R$-module and $(R_P)^f \cong (R^f)_P$, we obtain that $\text{Hom}_{R_P}((R_P)^f, R_P) \cong R_P$ as $R_P$-modules. Since $R_P$ is also $F$-finite, we have again by Proposition 3.10 that $F_{R_P}(E_{R_P}(k(P))) \cong E_{R_P}(k(P))$, where $k(P) \cong R_P/PR_P$; i.e., $R_P$ is weakly FPI. As $P$ was arbitrary, we obtain that $R$ is FPI by Proposition 3.3(c).
To prove the general case, first note that by parts (c) and (e) of Proposition 3.4 we may assume that \( R \) is complete. Let \( k \) be the residue field of \( R \). By the Cohen Structure Theorem, \( R \cong A/I \) where \( A = k[[T_1, \ldots, T_n]] \), \( T_1, \ldots, T_n \) are indeterminates, and \( I \) is an ideal of \( A \). Let \( \ell \) be the algebraic closure of \( k \), \( B = \ell[[T_1, \ldots, T_n]] \), and \( S = B/IB \). Note that as \( B \) is faithfully flat over \( A \) ([15, Theorem 22.4(i)]), \( S \) is faithfully flat over \( R \). Now suppose that \( R \) is weakly FPI. Then, by [6, Theorem 1], we have that \( E = R/P \) is local and complete. Let \( E \) be a weakly FPI ring. Then, by [6, Theorem 1], we have that \( E \sim R/P \). Hence, for all \( q \), we have \( E/R = E/P = 0 \), and \( \text{ht} P = 0 \). Therefore, \( E \) is a weakly FPI ring.

We next show that a weakly FPI ring has no embedded associated primes:

**Proposition 3.12.** Let \( R \) be a weakly FPI ring. Then \( R \) satisfies Serre’s condition \( S_1 \).

**Proof.** By Propositions 3.3(a) and 3.4(c) and [15, Theorem 23.9(iii)], we may assume that \( R \) is local and complete. Let \( P \in \text{Spec} R \) and \( s = \dim R/P \). Since \( \omega_{R/P} \) is a rank one torsion-free \( R/P \)-module, there exists an exact sequence \( 0 \longrightarrow \omega_{R/P} \longrightarrow R/P \). By Matlis duality, we obtain an exact sequence

\[
E_{R/P} \longrightarrow H^1_m(R/P) \longrightarrow 0,
\]

where \( E_{R/P} := E_{R/P}(R/m) \). Applying \( F^e_R \) to this sequence, we obtain the exactness of

\[
F^e_R(E_{R/P}) \longrightarrow H^s_m(R/P^{[q]}) \longrightarrow 0,
\]

where \( q = p^e \). By Proposition 2.3(a), \( \text{Ann}_R H^1_m(R/P^{[q]}) = U_R(R/P^{[q]}) \). Note that as \( \text{Min}_R R/P^{[q]} = \{ P \} \), \( U_R(R/P^{[q]}) = \psi^{-1}(P^{[q]}R_P) \), where \( \psi : R \longrightarrow R_P \) is the natural map. Hence, for all \( q = p^e \) we have

\[(#) \quad \text{Ann}_R F^e_R(E_{R/P}) \subseteq \psi^{-1}(P^{[q]}R_P).
\]

Now suppose that \( P \in \text{Ass}_R R \). Then there exists an exact sequence \( 0 \longrightarrow R/P \longrightarrow R \). Dualizing, we have \( E \longrightarrow E_{R/P} \longrightarrow 0 \) is exact, where \( E = E_{R}(R/m) \). Applying \( F^e_R \) and using that \( F^e_R(E) \cong E \), we have an exact sequence

\[
E \longrightarrow F^e_R(E_{R/P}) \longrightarrow 0.
\]

Dualizing again, we obtain an exact sequence

\[
0 \longrightarrow F^e_R(E_{R/P})^\vee \longrightarrow R.
\]

Note that as \( p E_{R/P} = 0 \), \( P^{[q]} F^e_R(E_{R/P}) = P^{[q]} F^e_R(E_{R/P})^\vee = 0 \) for all \( q = p^e \). Hence, for all \( q \) we have an exact sequence

\[
0 \longrightarrow F^e_R(E_{R/P})^\vee \longrightarrow H^1_P(R).
\]

Therefore, there exists a positive integer \( n \) such that \( P^n \subseteq \text{Ann}_R F^e_R(E_{R/P}) \) for all \( e \). By \((#)\), this implies \( P^n R_P \subseteq P^{[q]} R_P \) for all \( q = p^e \). Hence, \( P^n R_P = 0 \) and \( \text{ht} P = 0 \). \( \square \)

In general, if \( R \) is weakly FPI and \( x \) is a non-zero-divisor on \( R \) then \( R/(x) \) need not be weakly FPI. Otherwise, using Propositions 3.4(b) and 3.12 and [15, Exercise 18.1], one could prove that every weakly FPI ring is Gorenstein; but there exist quasi-Gorenstein rings (hence weakly FPI rings) which are not Gorenstein. However, it is true that if \((R, m)\) is a
local weakly FPI ring and $\text{Tor}^R_i(R^f, E) = 0$ then $R/(x)$ is weakly FPI for every non-zero-divisor $x \in m$. This can be proved by a simple modification of the arguments used in the proof of Theorem 3.14 below, where we provide a criterion for $R$ to be Gorenstein in terms of the modules $\text{Tor}^R_i(R^f, E)$. Before proving this result, we first need the following lemma:

**Lemma 3.13.** Let $(R, m)$ be a local ring and $I$ an ideal generated by a regular sequence. Then

$$R/I \otimes_R E_{R/I[x]}(R/m) \cong E_{R/I}(R/m).$$

*Proof.* Without loss of generality, we may assume $R$ is complete. Let $E = E_R(R/m)$. Note that $E_{R/I[x]}(R/m) \cong \text{Hom}_R(R/I[x], E)$ and $E_{R/I}(R/m) \cong \text{Hom}_R(R/I, E)$. Taking Matlis duals it suffices to prove that $\text{Hom}_R(R/I, R/I[x]) \cong R/I$. But this is easily seen to hold as $I$ is generated by a regular sequence. \qed

The following result is dual to Theorem 1.1 of Goto [7], which holds in the case where the Frobenius map is a finite morphism.

**Theorem 3.14.** Let $(R, m)$ be a local ring and $E = E_R(R/m)$. The following conditions are equivalent:

1. $\text{Tor}_0^R(R^f, E) \cong E$ and $\text{Tor}^R_i(R^f, E) = 0$ for all $i = 1, \ldots, \text{depth } R$
2. $R$ is Gorenstein.

*Proof.* Condition (2) implies (1) by Lemma 2.2 and Proposition 3.6 (note $E \cong H_R^d(R)$). Conversely, suppose condition (1) holds. Let $x = x_1, \ldots, x_r \in m$ be a maximal regular sequence on $R$ and $K(x)$ the Koszul complex with respect to $x$. Then $K(x) \xrightarrow{\epsilon} R/(x) \rightarrow 0$ is exact, where $\epsilon$ is the augmentation map. Dualizing, we have $0 \rightarrow E_{R/(x)}(R/m) \rightarrow K(x)^v$ is exact. Since $K(x)^v_j \cong E^{(j)}$ for all $j$ and $\text{Tor}^R_i(R^f, E) = 0$ for $1 \leq i \leq r$, we obtain that $0 \rightarrow F_R(E_{R/(x)}(R/m)) \rightarrow F_R(K(x)^v)$ is exact. In particular, since $F_R(E) \cong E$, we have an exact sequence

$$0 \rightarrow F_R(E_{R/(x)}(R/m)) \rightarrow E \xrightarrow{[x_1^{\ell}, \ldots, x_r^{\ell}]} E^r.$$

Hence,

$$F_R(E_{R/(x)}(R/m)) \cong \text{Hom}_R(R/(x)[\ell], E) \cong E_{R/(x)[\ell]}(R/m).$$

Using Lemma 3.13, we have

$$F_{R/(x)}(E_{R/(x)}(R/m)) \cong R/(x) \otimes_R F_R(E_{R/(x)}(R/m))$$

$$\cong R/(x) \otimes_R E_{R/(x)[\ell]}(R/m)$$

$$\cong E_{R/(x)}(R/m).$$

This says that $R/(x)$ is weakly FPI. Since $\text{depth } R/(x) = 0$, we must have $\dim R/(x) = 0$ by Proposition 3.12. But then $R/(x)$ is Gorenstein by Proposition 3.4(b). Hence, $R$ is Gorenstein. \qed
4. One-dimensional FPI rings

We now turn our attention to the one-dimensional case. If \( R \) is a local ring possessing an ideal which is also a canonical module of \( R \), this ideal is referred to as a canonical ideal of \( R \). If \( (R, m) \) is a one-dimensional Cohen-Macaulay local ring, then \( R \) has a canonical ideal (necessarily \( m \)-primary) if and only if \( \hat{R} \) is generically Gorenstein ([10, Satz 6.21]). The following result can viewed as a generalization of Lemma 2.6 of [7], which holds in the case where the Frobenius map is finite:

**Theorem 4.1.** Let \( (R, m) \) be a one-dimensional local ring. The following conditions are equivalent:

(a) \( R \) is weakly FPI;
(b) \( R \) is FPI;
(c) \( R \) is Cohen-Macaulay and has a canonical ideal \( \omega_R \) such that \( \omega_R \sim \omega_R^{[p]} \).

**Proof.** Since (b) trivially implies (a), it suffices to prove (a) implies (c) and (c) implies (b).

We first prove (a) implies (c): As \( R \) is weakly FPI, \( R \) is Cohen-Macaulay by Proposition 3.12. Furthermore, \( \hat{R} \) is weakly FPI and thus FPI by Theorem 3.11. Thus, \( \hat{R} \) is generically Gorenstein, which implies \( R \) possesses a canonical ideal \( \omega_R \). To show \( \omega_R \sim \omega_R^{[p]} \), it suffices to show that \( \omega_R^{[p]} \) is a canonical ideal of \( R \). Since \( \omega_R^{[p]} \) is a canonical ideal for \( R \) if and only if \( \omega_R^{[p]} \otimes_R \hat{R} \cong (\omega_R \hat{R})^{[p]} \) is a canonical ideal for \( \hat{R} \), we may assume without loss of generality that \( R \) is complete. Since \( R \) is Cohen-Macaulay, \( H^1_m(R) \cong E \), where \( E = E_R(R/m) \).

Applying local cohomology to the exact sequence

\[
0 \to \omega_R \to R \to R/\omega_R \to 0
\]

yields an exact sequence

\[
0 \to R/\omega_R \to E \to H^1_m(R) \to 0.
\]

Applying \( F_R \), we have an exact sequence

\[
0 \to R/\omega_R^{[p]} \to E \to H^1_m(R) \to 0,
\]

where we have used Proposition 2.1(e) and Lemma 2.2. Dualizing, we have an exact sequence

\[
0 \to \omega_R \to R \to \text{Hom}_R(R/\omega_R^{[p]}, E) \to 0.
\]

From the exactness of \( 0 \to \text{Hom}_R(R/m, R/\omega_R^{[p]}) \to \text{Hom}_R(R/m, E) \), we see that the socle of \( R/\omega_R^{[p]} \) is one-dimensional and hence \( R/\omega_R^{[p]} \) is Gorenstein. Thus, \( \text{Hom}_R(R/\omega_R^{[p]}, E) \cong R/\omega_R^{[p]} \) and we obtain an exact sequence

\[
0 \to \omega_R \to R \to R/\omega_R^{[p]} \to 0.
\]

This implies that \( \omega_R \cong \omega_R^{[p]} \).

Next, we prove (c) implies (b): Since \( R \) is Cohen-Macaulay and possesses a canonical ideal, \( R \) is a homomorphic image of a Gorenstein ring (see [2, Theorem 3.3.6]). Hence, by Theorem 3.11, it suffices to prove that \( R \) is weakly FPI. Let \( \pi : F_R(\omega_R) \to \omega_R^{[p]} \) be the natural surjection given by \( \pi(r \otimes u) = ru^p \) and let \( C = \ker \pi \). Since \( R_P \) is Gorenstein for all
primes $P \neq m$, $\dim C = 0$. Consequently, $H^1_m(C) = 0$ and $H^1_m(F_R(\omega_R)) \cong H^1_m(\omega_R^{[p]})$. Since $E \cong H^1_m(\omega_R)$ and $\omega_R \cong \omega_R^{[p]}$, we have

$$F_R(E) \cong F_R(H^1_m(\omega_R)) \cong H^1_m(F_R(\omega_R)) \cong H^1_m(\omega_R^{[p]}) \cong H^1_m(\omega_R) \cong E.$$ 

Hence, $R$ is weakly FPI. \hfill \qed

We remark that there exist one-dimensional local FPI rings which are not Gorenstein. In fact, the next result shows that every one-dimensional $F$-pure ring is FPI. Recall that a homomorphism $A \rightarrow B$ of commutative rings is called pure if the map $M \rightarrow B \otimes_A M$ is injective for every $A$-module $M$. A ring $R$ of prime characteristic is called $F$-pure if the Frobenius map $f : R \rightarrow R$ is pure.

**Proposition 4.2.** Let $R$ be a one-dimensional $F$-pure ring. Then $R$ is FPI.

**Proof.** By Proposition 3.3(b) and since $F$-purity localizes, we may assume that $R$ is local. By Theorem 4.1, it suffices to show that $R$ is weakly FPI. By Proposition 3.4(c) and [11, Corollary 6.13], we may assume $R$ is complete. Let $k$ be the residue field of $R$. By the Cohen Structure Theorem, $R \cong A/I$ where $A = k[[T_1, \ldots, T_n]]$, $T_1, \ldots, T_n$ are indeterminates, and $I$ is an ideal of $A$. Let $\ell$ be the algebraic closure $k$, $B = \ell[[T_1, \ldots, T_n]]$, and $S = B/IB$. Note that as $B$ is faithfully flat over $A$, $S$ is faithfully flat over $R$. Since $R$ is $F$-pure we have that $S$ is $F$-pure by Fedder’s criterion [4, Theorem 1.12]. Finally, by [6, Theorem 1], $E_S(\ell) \cong E_R(k) \otimes_R S$. Hence, $S$ is weakly FPI if and only if $R$ is weakly FPI. Thus, resetting notation, we may assume $R$ is complete and its residue field $k$ is algebraically closed. By [8, Theorem 1.1], $R \cong k[[T_1, \ldots, T_n]]/I$ where $I = \langle \{T_iT_j \mid 1 \leq i < j \leq n\} \rangle$. By Goto [7, Example 2.8], $\omega_R = (T_2 - T_1, \ldots, T_n - T_1)R$ is a canonical ideal of $R$, $T_1 + \cdots + T_n$ is a non-zerodivisor on $R$, and $\omega_R^{[p]} = (T_1 + \cdots + T_n)^{p-1}\omega_R$. Hence, $\omega_R^{[p]} \cong \omega_R$ and $R$ is weakly FPI by Theorem 4.1. \hfill \qed

As a specific example of a one-dimensional non-Gorenstein FPI ring, let $k$ be any field of characteristic $p$ and $R = k[[x,y,z]]/(xy,xz,yz)$. Then $R$ is a one-dimensional local ring which is $F$-pure (and hence FPI) but not Gorenstein. Notice in this example that $R$ has three associated primes. Regarding this we note the following, which is a consequence of Corollary 1.3 of [7]:

**Corollary 4.3.** Let $R$ be a one-dimensional complete local ring with algebraically closed residue field and suppose $R$ has at most two associated primes. The following are equivalent:

(a) $R$ is weakly FPI;

(b) $R$ is Gorenstein.

**Proof.** Notice that the hypotheses imply that $R$ is $F$-finite. Hence, (a) is equivalent to the condition that $\text{Hom}_R(R^f, R_R) \cong R_R$ by Corollary 3.10. The result now follows from [7, Corollary 1.3]. \hfill \qed

**References**


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