GORENSTEIN RINGS AND IRREDUCIBLE PARAMETER IDEALS

THOMAS MARLEY AND MARK W. ROGERS

Abstract. Given a Noetherian local ring \((R, m)\) it is shown that there exists an integer \(\ell\) such that \(R\) is Gorenstein if and only if some system of parameters contained in \(m^\ell\) generates an irreducible ideal. We obtain as a corollary that \(R\) is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal. It is also shown that, for a finitely generated \(R\)-module \(M\), a sequence \(x = x_1, \ldots, x_r\) of elements in \(m\) is \(M\)-regular if and only if the canonical map \(M/(x)M \rightarrow H^r_{(x)}(M)\) given by the direct limit is injective.

1. Introduction

It is well-known that a commutative Noetherian local ring \((R, m)\) is Gorenstein if and only if \(R\) is Cohen-Macaulay and some ideal generated by a system of parameters (called a parameter ideal) is irreducible. Perhaps less widely known is a result of Northcott and Rees which states that if every parameter ideal is irreducible then \(R\) is Cohen-Macaulay [NR, Theorem 1]. Hence, \(R\) is Gorenstein if and only if every parameter ideal is irreducible. There are, however, examples of non-Gorenstein rings possessing irreducible parameter ideals: \((y)R\) is irreducible in the local ring \(R = \mathbb{Q}[[x, y]]/(x^2, xy)\), for example. In a discussion of the Northcott-Rees result between the second author and William Heinzer, the following question arose: If \(R\) contains a system of parameters \(x_1, x_2, \ldots, x_d\) such that for every positive integer \(n\), the ideal \((x_1^n, x_2^n, \ldots, x_d^n)\) is irreducible, is \(R\) necessarily Gorenstein?

A concept related to this question was studied by Hochster: \(R\) is called approximately Gorenstein if every power of \(m\) contains an irreducible \(m\)-primary ideal. While approximately Gorenstein rings must have positive depth, they need not be Cohen-Macaulay. In fact, every complete Noetherian domain is approximately Gorenstein [Ho, Theorem 1.6]. However, our principal result (Theorem 2.9) shows that if a high enough power of \(m\) contains an irreducible parameter ideal then the ring is Cohen-Macaulay (and hence Gorenstein):

**Theorem:** Let \((R, m)\) be a Noetherian local ring. Then there exists an integer \(\ell\) such that \(R\) is Gorenstein if and only if some parameter ideal contained in \(m^\ell\) is irreducible.

As a consequence, a local ring \(R\) is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal, answering the question posed above. We also show that the integer \(\ell\) identified in this theorem may be taken to be the least integer

---

1991 Mathematics Subject Classification. Primary 13D45; Secondary 13H10.
Key words and phrases. Gorenstein, system of parameters, irreducible ideal.

The second author was supported for eight weeks during the summer of 2006 through the University of Nebraska-Lincoln’s **Mentoring through Critical Transition Points** grant (DMS-0354281) from the National Science Foundation.
\[ \delta = \delta(R) \text{ such that the canonical map} \]
\[ \text{Ext}_R^d(R/m^\delta, R) \rightarrow \lim \text{Ext}_R^d(R/m^n, R) \cong H_{m}^d(R) \]
is surjective after applying the functor \( \text{Hom}_R(R/m, -) \), where \( d = \dim R \).

We note that Theorem 2.9 is known in some special cases. Recent work by Goto-Sakurai [GSa1, GSa2], Liu-Rogers [LR], and Rogers [R] has shown that some rings having finite local cohomologies have eventual constant index of reducibility of parameter ideals, and these results may be used to prove our Main Theorem under additional hypotheses. (The index of reducibility of an ideal \( I \) is the number of irreducible ideals appearing in any irredundant expression of \( I \) as an intersection of irreducible ideals; in the case that \( R/I \) has finite length, the index of reducibility of \( I \) is the dimension of \( \text{Hom}_R(R/m, R/I) \) as an \( R/m \)-vector space, where \( m \) denotes the maximal ideal of \( R \).) To be precise, suppose \( R \) has finite local cohomologies (that is, the local cohomology modules \( H^i_m(R) \) have finite length when \( i \neq d \)) and assume that one of the following conditions holds: Either \( R \) is quasi-Buchsbaum (that is, \( m H^i_m(R) = 0 \) for \( i \neq d \)), or there is some integer \( t \) with \( 0 < t < d \) such that \( H^i_m(R) = 0 \) for all \( i \) with \( i \neq 0, t, d \). Then there is an integer \( \ell \) such that the index of reducibility of every parameter ideal contained in \( m^\ell \) is equal to \( \sum_{i=0}^{d-1} \binom{d}{i} \dim_R \text{Hom}_R(R/m, H^i_m(R)) \). (This expression for the eventual constant index of reducibility first appeared in [GSu].) Thus, if we further assume that every power of the maximal ideal contains an irreducible parameter ideal, then this eventual constant index of reducibility must be 1: i.e., \( \sum_{i=0}^{d} \binom{d}{i} \dim_R \text{Hom}_R(R/m, H^i_m(R)) = 1 \). Since \( H^d_m(R) \neq 0 \), we must have \( H^i_m(R) = 0 \) for \( i < d \). Thus, \( R \) is Cohen-Macaulay and hence Gorenstein.

There are two main ingredients to the proof of Theorem 2.9. The first is an interesting characterization of regular sequences in terms of direct limits (Proposition 2.3):

**Proposition:** Let \( R \) be a Noetherian ring, let \( x_1, \ldots, x_r \) be a sequence of elements in the Jacobson radical of \( R \), and let \( M \) be a finitely generated \( R \)-module. Consider the direct system \( \{ M/(x_1^n, \ldots, x_r^n)M \}_{n \geq 1} \) given by the maps
\[ M/(x_1^n, \ldots, x_r^n)M \xrightarrow{(x_1 \ldots x_r)^{t-s}} M/(x_1^t, \ldots, x_r^t)M \]
for \( 1 \leq s \leq t \). Then \( x_1, \ldots, x_r \) is a regular sequence on \( M \) if and only if the canonical map from \( M/(x_1, \ldots, x_r)M \) to the direct limit of this system is injective.

The proof of this proposition is reminiscent of an argument due to Hochster and Huneke that (under mild hypotheses) if some parameter ideal of \( R \) is tightly closed then \( R \) is Cohen-Macaulay. (See [HH] or [Hu], for example.)

The second ingredient is a result due to Goto and Sakurai [GSa1, Lemma 3.12] which states that if \( \delta = \delta(R) \) (as defined above), then for every parameter ideal \( (x_1, \ldots, x_d) \) contained in \( m^\delta \) the canonical map \( R/(x_1, \ldots, x_d) \rightarrow \lim \text{Ext}_R/R(x_1^n, \ldots, x_d^n) \) is surjective upon applying \( \text{Hom}_R(R/m, -) \). The proof of this proposition as given in [GSa2], while illuminating, is quite terse. Since this result is crucial to our paper, and indeed crucial for all recent research on the index of reducibility of parameter ideals, we give a more detailed proof in Section 3.

## 2. Main Results

As general references for terminology and well-known results, we refer the reader to [Mat] or [BH]. Throughout, \( R \) denotes a Noetherian ring. In case \( R \) is local with maximal ideal
Let $m$ and $M$ be an $R$-module, the socle of $M$ is defined to be $(0 :_M m) = \{x \in M \mid mx = 0\}$. The socle of $M$ is denoted by Soc$_R M$, or simply Soc $M$ if there is no confusion about the ring. It is clear that Soc$(\cdot)$ is a left exact covariant functor in a natural way. We often will identify this functor with the functor Hom$_R(R/m, \cdot)$.

**Definition 2.1.** Let $x_1, \ldots, x_r \in R$ and let $M$ be an $R$-module. Define

$$\{x_1, \ldots, x_r\}^\text{lim}_M := \bigcup_{n \geq 0} \{(x_1^{n+1}, \ldots, x_r^{n+1}) \cdot M : x_1^n \cdot x_r^n\}.$$ 

If $M = R$ we write $\{x_1, \ldots, x_r\}^\text{lim}$ for $\{x_1, \ldots, x_r\}^\text{lim}_R$.

We make the following remarks concerning this operation:

**Remark 2.2.** Let $I = (x_1, \ldots, x_r)$ be an ideal of $R$ and let $M$ be an $R$-module.

(a) The set $\{x_1, \ldots, x_r\}^\text{lim}_M$ is a submodule of $M$ containing $IM$.

(b) If $\{x_1, \ldots, x_r\}$ is a regular sequence on $M$ then $\{x_1, \ldots, x_r\}^\text{lim}_M = IM$. (cf. [Mat, Theorem 16.2]).

(c) Consider the direct system $\{M/(x_1^n, \ldots, x_r^n)M\}_{n \geq 1}$ given by the maps

$$M/(x_1^n, \ldots, x_r^n)M \xrightarrow{(x_1^{n-1} \cdots x_r^{n-1})} M/(x_1^n, \ldots, x_r^n)M$$

for $1 \leq s \leq t$. Then the kernel of the canonical map

$$\phi_t : M/(x_1^t, \ldots, x_r^t)M \rightarrow \lim_{\leftarrow} M/(x_1^n, \ldots, x_r^n)M \cong H^t_I(M)$$

is $\{x_1^t, \ldots, x_r^t\}^\text{lim}_M/(x_1^t, \ldots, x_r^t)M$. Hence by (b), if $x_1, \ldots, x_r$ is a regular sequence on $M$ then $\phi_t$ is injective for every $t$.

(d) If $R$ is local ring of prime characteristic and $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are two minimal generating sets for $I$ then $\{x_1, \ldots, x_r\}^\text{lim} = \{y_1, \ldots, y_r\}^\text{lim}$ [Hu, Remark 5.6]. In this context, $\{x_1, \ldots, x_r\}^\text{lim}$ is called the limit closure of $I$ and is denoted by $I^\text{lim}$.

(e) If $R$ is a local equidimensional ring of prime characteristic which is the homomorphic image of a Cohen-Macaulay ring and $x_1, \ldots, x_r$ are parameters (i.e., ht$(x_1, \ldots, x_r) = r$), then $(x_1, \ldots, x_r)^\text{lim} \subseteq I^*$, where $I^*$ denotes the tight closure of $I$ [Hu, Theorem 2.3(b)].

**Proposition 2.3.** Let $x_1, \ldots, x_r$ be elements in the Jacobson radical of $R$ and let $M$ be a finitely generated $R$-module. The following conditions are equivalent:

(a) $\{x_1, \ldots, x_r\}^\text{lim}_M = (x_1, \ldots, x_r)M$.

(b) $x_1, \ldots, x_r$ is a regular sequence on $M$.

Proof. By Remark 2.2(b), (b) implies (a). We prove that (a) implies (b) by induction on $r$. In the case $r = 1$, let $x = x_1$. Suppose $\{x\}^\text{lim}_M = (x)M$ and $xx = 0$ for some $\alpha \in M$. We claim that $\alpha \in (x^k)M$ for all $k \geq 0$. This is clearly true for $k = 0$, so suppose $\alpha = x^k \beta$ for some $k \geq 0$ and $\beta \in M$. Then $x^{k+1} \beta = 0$, and thus $\beta \in \{x\}^\text{lim}_M = (x)M$. Hence, $\alpha \in (x^{k+1})M$. As $x$ is in the Jacobson radical and $M$ is finitely generated, $\cap_h(x^k)M = 0$ by Krull’s Intersection Theorem. Hence, $\alpha = 0$ and $x$ is a non-zero-divisor on $M$.

Suppose now that $r \geq 1$. To complete the proof, we will show the following:

1. $\{x_1, \ldots, x_{r-1}\}^\text{lim}_M = (x_1, \ldots, x_{r-1})M$.
2. $x_r$ is a non-zero-divisor on $M/(x_1, \ldots, x_{r-1})M$. 


Item (1) will allow us to use the inductive hypothesis to conclude that $x_1, \ldots, x_{r-1}$ is a regular sequence on $M$.

To prove (1), let $\alpha \in \{x_1, \ldots, x_{r-1}\}_M^{\lim}$. We claim that for all $k \geq 0$, $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^k)M$. Again by Krull’s Intersection Theorem, this will imply that $\alpha \in (x_1, \ldots, x_{r-1})M$. The case $k = 0$ is clear, so suppose $\alpha = \omega + x_r^k \beta$ where $\omega \in (x_1, \ldots, x_{r-1})M$ and $\beta \in M$. Thus, $x_r^k \beta \in \{x_1, \ldots, x_{r-1}\}_M^{\lim}$. Hence, there exists $t \geq 0$ such that

$$\left(x_1 \cdots x_{r-1}\right)^t x_r^k \beta \in \left(x_1^{t+1}, \ldots, x_{r-1}^{t+1}\right)M.$$ 

Multiplying by $(x_1 \cdots x_{r-1})^k x_r^t$, we obtain

$$\left(x_1 \cdots x_r\right)^{t+k} \beta \in \left(x_1^{t+k+1}, \ldots, x_{r-1}^{t+k+1}\right)M \subseteq \left(x_1^{t+k+1}, \ldots, x_r^{t+k+1}\right)M.$$ 

Hence, $\beta \in \{x_1, \ldots, x_r\}_M^{\lim} = (x_1, \ldots, x_r)M$. Thus, $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^{k+1})M$.

The proof of (2) is similar: Suppose $x_r \alpha \in (x_1, \ldots, x_{r-1})M$ for some $\alpha \in M$. We claim that $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^k)M$ for all $k \geq 0$. Suppose $\alpha = \omega + x_r^k \beta$ where $\omega \in (x_1, \ldots, x_{r-1})M$ and $\beta \in M$. Then $x_r \alpha = x_r \omega + x_r^{k+1} \beta$. Hence, $x_r^{k+1} \beta \in (x_1, \ldots, x_{r-1})M$.

Multiplying by $(x_1 \cdots x_{r-1})^k x_r^t$, we obtain that

$$\left(x_1 \cdots x_r\right)^{k+1} \beta \in \left(x_1^{k+2}, \ldots, x_{r-1}^{k+2}\right)M \subseteq \left(x_1^{k+2}, \ldots, x_r^{k+2}\right)M.$$ 

Hence, $\beta \in \{x_1, \ldots, x_r\}_M^{\lim} = (x_1, \ldots, x_r)M$ and $\alpha \in (x_1, \ldots, x_{r-1})M + (x_r^{k+1})M$. 

\textbf{Corollary 2.4.} Let $x_1, \ldots, x_r$ elements of the Jacobson radical of $R$ and let $M$ be a finitely generated $R$-module. For each $t \geq 1$ let $\phi_t$ be the canonical map $M/(x_1^t, \ldots, x_r^t)M \to \bigcap_{r=1}^t M$. The following conditions are equivalent:

(a) $\{x_1, \ldots, x_r\}$ is a regular sequence on $M$
(b) $\phi_t$ is injective for some $t \geq 1$.
(c) $\phi_t$ is injective for all $t \geq 1$.

We note that Proposition 2.3 gives another proof of the well-known result that F-rational rings are Cohen-Macaulay (e.g., [HH] or [Hu, Theorem 8.2]):

\textbf{Corollary 2.5.} Let $R$ be a equidimensional local ring of prime characteristic and which is the homomorphic image of a Cohen-Macaulay ring. Suppose $(x_1, \ldots, x_d)^* = (x_1, \ldots, x_d)$ for some system of parameters $x_1, \ldots, x_d$ of $R$. Then $R$ is Cohen-Macaulay.

\textbf{Proof.} By Remark 2.2(d) we have

$$(x_1, \ldots, x_d) \subseteq (x_1, \ldots, x_d)^{\lim} \subseteq (x_1, \ldots, x_d)^*.$$ 

The result now follows from Proposition 2.3. 

In the sequel we adopt the following notation: For a sequence of elements $\mathbf{x} = x_1, \ldots, x_r$ and $t \in \mathbb{N}$ we let $\mathbf{x}^t$ denote the sequence $x_1^t, \ldots, x_r^t$. For $x \in R$ we let $K(x)$ denote the Koszul chain complex $0 \to R \xrightarrow{x} R \to 0$, where the first $R$ is in homological degree 1. For the sequence $\mathbf{x}$ the Koszul chain complex $K(\mathbf{x})$ is defined to be the chain complex $K(x_1) \otimes \cdots \otimes K(x_r)$. For $1 \leq s \leq t$ there exists a chain maps $\phi_s^t: K(\mathbf{x}^s) \to K(\mathbf{x}^t)$ given by
If \( \text{lim} \) is the direct limit, then the map \( \phi_s \) is the chain map

\[
\begin{array}{cccc}
K(x^t) & : & 0 & \to R & \xrightarrow{x^t} & R & \xrightarrow{x^{t-s}} & R & \to 0 \\
\downarrow \phi_s(x) & & & & & & \downarrow \phi_s(x) & \\
K(x^s) & : & 0 & \to R & \xrightarrow{x^s} & R & \to 0
\end{array}
\]

For an \( R \)-module \( M \), the \( i \)th Koszul cohomology of \( M \) with respect to \( x \), denoted \( H_i(x; M) \), is the \( i \)th cohomology of \( \text{Hom}_R(K(x), M) \). The maps \( \phi_s \) above induce chain maps

\[
\text{Hom}_R(K(x^s), M) \to \text{Hom}_R(K(x^t), M)
\]

for all \( 1 \leq s \leq t \). By [Gr, Theorem 2.8], we have \( \lim H_i(x^n; M) \cong H_i(x)(M) \).

We make the following elementary observations concerning direct limits:

**Remark 2.6.** Let \( \{M_n, \lambda^n_s\} \) be a direct system of \( R \)-modules over a directed index set and let \( \phi_t : M_t \to \lim M_n \) be the canonical maps given by the definition of the direct limit.

(a) If \( \lim M_n \) is finitely generated then \( \phi_t \) is surjective for all sufficiently large \( t \).

(b) If \( A \) is a finitely presented \( R \)-module then \( \text{Hom}_R(A, \lim M_n) \cong \lim \text{Hom}_R(A, M_n) \).

**Proof.** Part (a) is an easy consequence of [Rot, Theorem 2.17]. For part (b), see Exercise 26, Chapter III of [L]. \( \square \)

**Definition 2.7.** Let \( (R, m) \) be a local ring, let \( M \) be a finitely generated \( R \)-module, and let \( i \geq 0 \). By applying \( \text{Ext}_R^i(-, M) \) to the system of surjections

\[
\cdots \to R/m^3 \to R/m^2 \to R/m
\]

we obtain a direct system whose limit is \( \lim \text{Ext}_R^i(R/m^n, M) \cong H_m^i(M) \). By Remark 2.6(b),

\[
\lim \text{Soc Ext}_R^i(R/m^n, M) \cong \text{Soc} \lim \text{Ext}_R^i(R/m^n, M) \cong \text{Soc} H_m^i(M).
\]

Since \( H_m^i(M) \) is Artinian, \( \text{Soc} H_m^i(M) \) is finitely generated. Hence, by Remark 2.6(a) there exists a smallest nonnegative integer \( \ell_i(M) \) such that the map \( \text{Soc Ext}_R^i(R/m^t, M) \to \text{Soc} H_m^i(M) \) is surjective for all \( t \geq \ell_i(M) \).

The following proposition is essentially [GSa1, Lemma 3.12]. A complete proof is given in Section 3.

**Proposition 2.8.** [GSa1, Lemma 3.12] Let \( (R, m) \) be a Noetherian local ring and let \( M \) be a finitely generated \( R \)-module. For \( i \geq 0 \) and all \( m \)-primary ideals \( q = (x_1, \ldots, x_r) = (x) \) contained in \( m^{\ell_i(M)} \) the map

\[
\text{Soc} H_i(x; M) \to \text{Soc} H_m^i(M)
\]

induced by the canonical map \( H_i(x; M) \to \lim H_i(x^n; M) \) is surjective.

We now proceed with the proof of our main result:

**Theorem 2.9.** Let \( (R, m) \) be a Noetherian local ring of dimension \( d \) and let \( \ell = \ell_d(R) \). Then \( R \) is Gorenstein if and only if some parameter ideal contained in \( m^\ell \) is irreducible.
Proof. It suffices to show that if there exists a system of parameters $\mathbf{x} = x_1, \ldots, x_d$ contained in $m^t$ which generates an irreducible ideal then $R$ is Cohen-Macaulay (and hence Gorenstein). Let $\phi = \phi_1$ denote the canonical homomorphism from $H^d(\mathbf{x}; R) \cong R/(\mathbf{x})$ to $\varprojlim H^d(\mathbf{x}^t; R) \cong H^d_m(R)$. By Remark 2.2(c) we have an exact sequence

$$0 \to \frac{\{\mathbf{x}\}^\lim}{(\mathbf{x})} \to \frac{R}{(\mathbf{x})} \xrightarrow{\phi} H^d_m(R).$$

Applying the socle functor and using Proposition 2.8 we obtain the exact sequence

$$0 \to \text{Soc} \frac{\{\mathbf{x}\}^\lim}{(\mathbf{x})} \to \text{Soc} \frac{R}{(\mathbf{x})} \to \text{Soc} H^d_m(R) \to 0.$$

Since $H^d_m(R)$ is a nonzero Artinian module, it has a nonzero socle. Since $(\mathbf{x})$ is irreducible, $R/(\mathbf{x})$ has a one-dimensional socle. Hence, $\text{Soc}(\{\mathbf{x}\}^\lim/(\mathbf{x})) = 0$, which implies $\{\mathbf{x}\}^\lim = (\mathbf{x})$. By Proposition 2.3, we see that $\mathbf{x}$ is a regular and hence $R$ is Cohen-Macaulay. \(\square\)

Corollary 2.10. Let $(R, m)$ be a Noetherian local ring. Then $R$ is Gorenstein if and only if every power of the maximal ideal contains an irreducible parameter ideal.

Proof. Immediate from Theorem 2.9. \(\square\)

3. A Proof of Proposition 2.8

Throughout, $R$ denotes a Noetherian ring. We begin with a lemma:

Lemma 3.1. Let $\mathbf{x} = x_1, \ldots, x_r$ be a sequence of elements from $R$ and let $I = (\mathbf{x})R$. Then there exist a family of complexes $\{F(t)\}_{t \geq 1}$ and chain maps $\alpha(t): K(\mathbf{x}^t) \to F(t)$ and $\beta(t+1): F(t+1) \to F(t)$ such that for each $t \geq 1$

(1) $F(t)$ is a free resolution of $R/I^t$ and each $F(t)_i$ is finitely generated;
(2) $F(t)_0 = R$;
(3) $\alpha(t)_0$ and $\beta(t+1)_0$ are the identity maps;
(4) the diagram

$$\begin{array}{ccc}
K(\mathbf{x}^{t+1}) & \xrightarrow{\phi(t+1)} & K(\mathbf{x}^t) \\
\downarrow{\alpha(t+1)} & & \downarrow{\alpha(t)} \\
F(t+1) & \xrightarrow{\beta(t+1)} & F(t)
\end{array}$$

commutes, where $\phi(t+1)$ is the chain map $\phi_{t+1}^t$ defined in Section 2.

Proof. We use induction on $t$. Choose $F(1)$ to be any minimal free resolution of $R/I$ and $\alpha(1): K(\mathbf{x}) \to F(1)$ any lifting of $id_{R/I}: H_0(\mathbf{x}) \to H_0(F(1))$. Suppose $t \geq 1$ and for all $1 \leq k \leq t$ there exists resolutions $F(k)$ and chain maps $\alpha(k)$ and $\beta(k)$ which have the desired properties. We will construct $F(t+1)$, $\alpha(t+1)$ and $\beta(t+1)$. First, we simplify notation: Let $G := F(t)$, $C := K(\mathbf{x}^{t+1})$, and $\gamma = \alpha(t)\phi(t+1)$. We need to construct a resolution $F = F(t+1)$ of $R/I^t$ and chain maps $\alpha = \alpha(t+1): C \to F$ and $\beta = \beta(t+1): F \to G$ such that $\gamma = \alpha\beta$. The proof of this is a variation on the Horseshoe Lemma [Rot, Lemma 6.20]. Let $F_0 = R$ and $\beta_0 = \alpha_0 = id_R$. Suppose for some $k \geq 0$ there exists a commutative
diagram of the form

\[
\begin{array}{ccccccccc}
C_{k+1} & \overset{\partial_{k+1}}{\longrightarrow} & C_k & \overset{\partial_k}{\longrightarrow} & C_{k-1} & \longrightarrow & \cdots & \overset{\partial_1}{\longrightarrow} & C_0 & \overset{\partial_0}{\longrightarrow} & R/(x^{t+1}) & \longrightarrow & 0 \\
\downarrow{\alpha_k} & & \downarrow{\alpha_{k-1}} & & \downarrow{\alpha_0} & & \downarrow{\alpha_{-1}} & & \downarrow{} & & \downarrow{} & & \\
F_k & \overset{\partial'_k}{\longrightarrow} & F_{k-1} & \longrightarrow & \cdots & \overset{\partial'_1}{\longrightarrow} & F_0 & \overset{\partial'_0}{\longrightarrow} & R/I^{t+1} & \longrightarrow & 0 \\
\downarrow{\beta_k} & & \downarrow{\beta_{k-1}} & & \downarrow{\beta_0} & & \downarrow{\beta_{-1}} & & \downarrow{} & & \downarrow{} & & \\
G_k & \overset{\partial''_k}{\longrightarrow} & G_{k-1} & \longrightarrow & \cdots & \overset{\partial''_1}{\longrightarrow} & G_0 & \overset{\partial''_0}{\longrightarrow} & R/I^t & \longrightarrow & 0
\end{array}
\]

where the middle row is exact and \( F_i \) is a finitely generated free module and \( \gamma_i = \alpha_i \beta_i \) for all \( i \leq k \). (In the diagram, \( \alpha_{-1} \) and \( \beta_{-1} \) denote the natural surjections.) Let \( u_1, \ldots, u_s \in F_k \) be generators for \( \ker \partial_k \) and \( w_1, \ldots, w_z \in G_{k+1} \) be generators for \( \ker \partial'_{k+1} \). Let \( F_{k+1} \) be a free \( R \)-module of rank \( s + z \) and \( a_1, \ldots, a_s, b_1, \ldots, b_z \) a basis for \( F_{k+1} \). Define \( \partial'_{k+1}: F_{k+1} \rightarrow F_k \) by \( \partial'_{k+1}(a_i) = u_i \) for \( 1 \leq i \leq s \) and \( \partial'_{k+1}(b_i) = 0 \) for \( 1 \leq i \leq z \). Clearly, \( \text{im} \partial'_{k+1} = \ker \partial'_k \). Choose \( c_1, \ldots, c_s \in G_{k+1} \) such that \( \partial''_{k+1}(c_i) = \beta(k(u_i)) = \beta_k(\partial'_{k+1}(a_i)) \) for \( 1 \leq i \leq s \). Define \( \beta_{k+1}: F_{k+1} \rightarrow G_{k+1} \) by \( \beta_{k+1}(a_i) = c_i \) for \( 1 \leq i \leq s \) and \( \beta_{k+1}(b_i) = w_i \) for \( 1 \leq i \leq z \). Evidently, \( \partial''_{k+1} \beta_{k+1} = \beta_k \partial'_{k+1} \). Now let \( e_1, \ldots, e_p \) be a basis for \( C_{k+1} \). Choose \( f_1, \ldots, f_p \in F_{k+1} \) such that \( \partial''_{k+1}(f_i) = \alpha_k \partial_{k+1}(e_i) \) for \( 1 \leq i \leq p \). Then for \( 1 \leq i \leq p \),

\[
\partial''_{k+1} \gamma_{k+1}(e_i) = \gamma_{k} \partial_{k+1}(e_i) \\
= \beta_k (\gamma_{k} \partial_{k+1}(e_i)) \\
= \beta_k (\beta_k (f_i)) \\
= \partial''_{k+1} \beta_{k+1}(f_i).
\]

Hence, \( \gamma_{k+1}(e_i) - \beta_{k+1}(f_i) \in \ker \partial''_{k+1} \) for \( 1 \leq i \leq p \). For \( 1 \leq i \leq p \) let \( v_i \in Rb_1 + \cdots + Rb_z \) be such that \( \beta_{k+1}(v_i) = \gamma_{k+1}(e_i) - \beta_{k+1}(f_i) \). Finally, define \( \alpha_{k+1}: C_{k+1} \rightarrow F_{k+1} \) by \( \alpha_{k+1}(e_i) = f_i + v_i \) for \( 1 \leq i \leq p \). It is easily verified that \( \partial'_{k+1} \alpha_{k+1} = \alpha_k \partial_{k+1} \) and \( \gamma_{k+1} = \beta_{k+1} \alpha_{k+1} \). □

Let \( N \) be an arbitrary \( R \)-module, let \( x = x_1, \ldots, x_r \), and let \( I = (x) \) as in Lemma 3.1. Applying \( \text{Hom}_R(\cdot, N) \) to the commutative diagram in part (4) of this lemma, we get for all \( t \geq 1 \) a commutative square of cochain complexes

\[
\begin{array}{ccc}
\text{Hom}_R(F(t), N) & \overset{\beta(t+1,N)}{\longrightarrow} & \text{Hom}_R(F(t+1), N) \\
\alpha(t,N) & & \alpha(t+1,N) \\
\downarrow & & \downarrow \\
\text{Hom}_R(K(x^t), N) & \overset{\phi(t+1,N)}{\longrightarrow} & \text{Hom}_R(K(x^{t+1}), N).
\end{array}
\]

Taking \( i \)th cohomologies, we have for all \( t \geq 1 \) the commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^i(R/I^t, N) & \overset{\beta^i(t+1,N)}{\longrightarrow} & \text{Ext}^i(R/I^{t+1}, N) \\
\alpha^i(t,N) & & \alpha^i(t+1,N) \\
\downarrow & & \downarrow \\
\text{H}^i(x^t; N) & \overset{\phi^i(t+1,N)}{\longrightarrow} & \text{H}^i(x^{t+1}; N) \\
\text{lim} \downarrow & & \text{lim} \downarrow \\
\text{lim} \alpha^i(N) & & \text{lim} \alpha^i(N)
\end{array}
\]

\[
(3.1)
\]

\[\text{Lemma 3.2. For all } i \geq 0 \text{ the map } \text{lim} \alpha^i(N) \text{ of Diagram (3.1) is an isomorphism.}\]
Proof. For \( i = 0 \), we have the diagram
\[
\begin{array}{c}
\text{Hom}_R(R/I^t, N) \xrightarrow{\alpha^0_t(N)} \text{Hom}_R(R/I^{t+1}, N) \\
\alpha^0(N) \downarrow \quad \quad \downarrow \alpha^0_{t+1}(N)
\end{array}
\]
\[
\text{H}^0(x^t; N) \xrightarrow{\phi^0_{t+1}(N)} \text{H}^0(x^{t+1}; N).
\]
In this case, \( \alpha^0_t(N) \) is injective for all \( t \), so \( \lim \alpha^0_t(N) \) is injective, and since \( \{P^n\} \) and \( \{(x^n)\} \) are cofinal, the map \( \lim \alpha^0_t(N) \) is surjective.

Suppose \( i \geq 1 \) and \( \lim \alpha^i_t(M) \) is an isomorphism for all \( 0 \leq j \leq i - 1 \) and all \( R \)-modules \( M \). Let \( E \) be the injective hull of \( N \) and consider the short exact sequence
\[
0 \to N \to E \to C \to 0
\]
where \( C = E/N \). Since \( F(t) \) and \( K(x^t) \) are complexes of free modules, we have for all \( t \geq 1 \) a commutative diagram
\[
\begin{array}{c}
0 \quad \quad \text{Hom}_R(F(t), N) \xrightarrow{\alpha(t,N)} \text{Hom}_R(F(t), E) \xrightarrow{\alpha(t,E)} \text{Hom}_R(F(t), C) \xrightarrow{\alpha(t,C)} 0
\end{array}
\]
where the rows are short exact sequences of cochain complexes. We also note that \( \beta(t + 1) \) and \( \phi(t + 1) \) induce maps from this diagram to the same diagram but with \( t \) replaced everywhere by \( t + 1 \). (This would be represented by a 3-dimensional commutative diagram.)

Applying the long exact sequence on cohomology to this diagram, we obtain for \( t \geq 1 \) a commutative diagram with exact rows
\[
\begin{array}{c}
\text{Ext}^{i-1}_R(R/I^t, E) \xrightarrow{\alpha^{i-1}_t(E)} \text{Ext}^{i-1}_R(R/I^t, C) \xrightarrow{\alpha^{i-1}_t(C)} \text{Ext}^{i-1}_R(R/I^t, N) \xrightarrow{\alpha^i_t(N)} \text{Ext}^i(R/I^t, E)
\end{array}
\]
\[
\begin{array}{c}
\text{H}^{i-1}(x^t; E) \xrightarrow{\alpha^{i-1}_t(E)} \text{H}^{i-1}(x^t; C) \xrightarrow{\alpha^{i-1}_t(C)} \text{H}^{i-1}(x^t; N) \xrightarrow{\alpha^i_t(N)} \text{H}^i(x^t; E).
\end{array}
\]
Taking direct limits of these diagrams and using that \( \lim \text{H}^i(x^n; E) = 0 \) for all \( i \geq 1 \) ([Gr, Proposition 2.6]), we obtain a commutative diagram with exact rows
\[
\begin{array}{c}
\lim \text{Ext}^{i-1}_R(R/I^n, E) \xrightarrow{\lim \alpha^{i-1}_n(E)} \lim \text{Ext}^{i-1}_R(R/I^n, C) \xrightarrow{\lim \alpha^{i-1}_n(C)} \lim \text{Ext}^i(R/I^n, N) \xrightarrow{\lim \alpha^i_n(N)} \lim \text{H}^i(x^n; E)
\end{array}
\]
\[
\begin{array}{c}
\lim \text{H}^{i-1}(x^n; E) \xrightarrow{\lim \alpha^{i-1}_n(E)} \lim \text{H}^{i-1}(x^n; C) \xrightarrow{\lim \alpha^{i-1}_n(C)} \lim \text{H}^i(x^n; N) \xrightarrow{\lim \alpha^i_n(N)} 0.
\end{array}
\]
Since \( \lim \alpha^{i-1}_n(E) \) and \( \lim \alpha^{i-1}_n(C) \) are isomorphisms (by the induction hypothesis), we conclude that \( \lim \alpha^i_n(N) \) is an isomorphism. \( \square \)

We now proceed with:

Proof of Proposition 2.8: With the notation as in the statement of the Proposition, fix \( i \geq 0 \) and let \( \ell = \ell_i(M) \). Let \( q = (x) \) be a parameter ideal contained in \( m^\ell \). By applying the
socle functor to Diagram (3.1) (with $q$ in place of $I$) we obtain the commutative diagram

$$
\begin{array}{ccc}
\text{Soc Ext}^i_R(R/q, M) & \longrightarrow & \text{Soc} \lim\limits_{\rightarrow} \text{Ext}^i_R(R/q^n, M) \\
\text{Soc} \alpha^i(M) & \longrightarrow & \text{Soc} \lim\limits_{\rightarrow} \alpha^i(M) \\
\text{Soc} H^i(x; M) & \longrightarrow & \text{Soc} \lim\limits_{\rightarrow} H^i(x^n; M).
\end{array}
$$

By Lemma 3.2, the right arrow is an isomorphism. Let $J$ be an injective resolution of $M$. For each $t \geq 1$ the commutative square of natural surjections

$$
\begin{array}{ccc}
R/q^{t+1} & \longrightarrow & R/m^{\ell(t+1)} \\
\downarrow & & \downarrow \\
R/q^t & \longrightarrow & R/m^{\ell t}
\end{array}
$$

induces the commutative diagram of cochain complexes

$$
\begin{array}{ccc}
\text{Hom}_R(R/m^{\ell t}, J) & \longrightarrow & \text{Hom}_R(R/q^t, J) \\
\downarrow & & \downarrow \\
\text{Hom}_R(R/m^{\ell(t+1)}, J) & \longrightarrow & \text{Hom}_R(R/q^{t+1}, J) \\
\downarrow & & \downarrow \\
\lim\limits_{\rightarrow} \text{Hom}_R(R/m^{\ell n}, J) & \longrightarrow & \lim\limits_{\rightarrow} \text{Hom}_R(R/q^n, J).
\end{array}
$$

The map $\lim\limits_{\rightarrow} \varphi_n$ is an isomorphism. Indeed, since the maps in Diagram (3.3) are surjections, the maps in the top square of Diagram (3.4) are injections, and thus $\lim\limits_{\rightarrow} \varphi_n$ is an injection. Also, since $q$ is $m$-primary, $\{q^n\}$ and $\{m^{\ell n}\}$ are cofinal. Hence, $\lim\limits_{\rightarrow} \varphi_n$ is surjective.

Taking $ith$ cohomologies, applying the socle functor, and using Remark 2.6(b), we obtain the commutative diagram

$$
\begin{array}{ccc}
\text{Soc Ext}^i_R(R/m^{\ell}, M) & \longrightarrow & \text{Soc Ext}^i_R(R/q, M) \\
\downarrow & & \downarrow \\
\lim\limits_{\rightarrow} \text{Soc Ext}^i_R(R/m^{\ell n}, M) & \longrightarrow & \lim\limits_{\rightarrow} \text{Soc Ext}^i_R(R/q^n, M).
\end{array}
$$

Now, the left vertical arrow is surjective by the definition of $\ell$ and [Rot, Exercise 2.43]. As the bottom arrow is an isomorphism, we conclude that the right arrow is also surjective. We complete the proof of the proposition by noting that since the top and right maps in Diagram (3.2) are surjective, so is the bottom map.

**Acknowledgment:** The authors would like to thank Craig Huneke for offering an important insight to the proof of Proposition 2.3, and also William Heinzer and Jung-Chen Liu for careful readings of this manuscript.

**References**


Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA
E-mail address: tmarley@math.unl.edu
URL: http://www.math.unl.edu/~tmarley1

Department of Mathematics, Missouri State University, Springfield, MO 65897, USA
E-mail address: markrogers@missouristate.edu